

Dynamics of (Pseudo) Automorphisms of 3-space: Periodicity versus positive entropy

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§0. Introduction. We consider the family of birational maps of 3-space which may be written in affine coordinates as

$$f_{\alpha,\beta} : (x_1, x_2, x_3) \mapsto \left(x_2, x_3, \frac{\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3} \right). \quad (0.1)$$

The algebraic iterates $f_{\alpha,\beta}^n := f_{\alpha,\beta} \circ \cdots \circ f_{\alpha,\beta}$ are rational maps for all $n \in \mathbf{Z}$. Here we study the dynamics of $f = f_{\alpha,\beta}$, by which we mean the behavior of f^n as $n \rightarrow \pm\infty$. We have invertible dynamics since f has a rational inverse, but it does not behave like a diffeomorphism (or even a homeomorphism). There are two difficulties if we want to regard f as a mapping of points. First, there is the set of indeterminacy $\mathcal{I}(f)$; f blows up each point of $\mathcal{I}(f)$ to a variety of positive dimension. Second, there can be hypersurfaces E which are exceptional, in the sense that the codimension of $f(E - \mathcal{I}(f))$ is at least 2. We will say that f is a pseudo-automorphism if neither f nor f^{-1} has an exceptional hypersurface. In dimension 2, every pseudo-automorphism is in fact an automorphism. However, for pseudo-automorphisms, indeterminate behaviors are possible in higher dimension which have no analogue in dimension 2.

Given a rational map $f : X \dashrightarrow X$ there is a well-defined pullback map on cohomology, $f^* : H^*(X) \rightarrow H^*(X)$. Passage to cohomology, however, may not be compatible with iteration because the identity $(f^*)^n = (f^n)^*$ may not be valid. Given a birational map f in dimension 2, Diller and Favre [DiF] showed that there is a new manifold $\pi : Y \rightarrow X$ such that the iterates of the induced map f_Y behave naturally on cohomology, in the sense that $(f_Y^*)^n = (f_Y^n)^*$. In dimension greater than 2, however, no such theorem is known.

Given a rational map of \mathbf{P}^n we may consider modifications $\pi : X \rightarrow \mathbf{P}^n$, where π is a morphism which is birational. This induces a rational map $f_X := \pi^{-1} \circ f \circ \pi$ of X , which might have pointwise properties which are different from those of the original f . If f_X is a pseudo-automorphism, then f_X acts naturally on $H^{1,1}(X)$. The exponential rate of growth of f^n on $H^{p,p}$: $\delta_p(f) := \lim_{n \rightarrow \infty} \|f^{n*}|_{H^{p,p}(X)}\|^{1/n}$ is known as the p th dynamical degree and is a birational invariant (see [DS]).

Within the family (0.1) we find the first known examples of pseudo-automorphisms of positive entropy on blowups of \mathbf{P}^3 :

Theorem 1. *Suppose that $\alpha = (a, 0, \omega, 1)$ and $\beta = (0, 1, 0, 0)$ where $a \in \mathbf{C} \setminus \{0\}$ and ω is a non-real cube root of the unity. Then there is a modification $\pi : Z \rightarrow \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. The dynamical degrees $\delta_1(f) = \delta_2(f) \cong 1.28064 > 1$ are equal and are given by the largest root of $t^8 - t^5 - t^4 - t^3 + 1$. The entropy of f_Z is the logarithm of the dynamical degree and is thus positive.*

Theorem 2. *For the mappings in Theorem 1, there is a 1-parameter family of surfaces $S_c \subset Z$, $c \in \mathbf{C}$ which have the invariance $fS_c = S_{\omega c}$. For generic c , S_c is K3, and the restriction $f^3|_{S_c}$*

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is an automorphism. For generic c and c' , the surfaces S_c and $S_{c'}$ are biholomorphically inequivalent, and the automorphisms $f^3|_{S_c}$ and $f^3|_{S_{c'}}$ are not smoothly conjugate.

The surface S_0 is invariant, and the restriction f_{S_0} is an automorphism which has the same entropy as f . This is smaller than the entropy of the automorphism constructed in [M2, Theorem 1.2] and is thus the smallest known entropy for a *projective* $K3$ surface automorphism.

Closely related to the dynamics of f_Z is the $(1,1)$ -current T^+ which is expanded by f_Z^* , and a current T^- for f_Z^{-1} . This is obtained in §7, as well as the invariant $(2,2)$ -current $T^+ \wedge T^-$. The slices of T^\pm and $T^+ \wedge T^-$ on the surfaces S_c give the expanded/contracted currents, as well as the unique invariant measure, for the automorphism $f|_{S_c}$.

The following mappings have quadratic degree growth and complete integrability:

Theorem 3. *Suppose that $\beta = (0, 1, 0, 0)$ and either $\alpha = (0, 0, \omega, 1)$ or $\alpha = (a, 0, 1, 1)$ where $a \in \mathbf{C} \setminus \{1\}$, $\omega \neq 1$, and $\omega^3 = 1$. Then the degree of f^n grows quadratically in n . Further, there is a modification $\pi : Z \rightarrow \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. There is a two-parameter family of surfaces S_c , $c = (c_1, c_2) \in \mathbf{C}^2$ which are invariant under f^3 . For generic c and c' , S_c is a smooth $K3$ surface, and $S_c \cap S_{c'}$ is a smooth elliptic curve.*

For the mappings in Theorems 1 and 3, f is reversible on the level of cohomology: f_Z^* is conjugate to $(f_Z^{-1})^* = (f_Z^*)^{-1}$. The identity $\delta_1(f) = \delta_2(f)$ for such maps is a consequence of the duality between $H^{1,1}$ and $H^{2,2}$, so they are not cohomologically hyperbolic, in the terminology of [G2]. For each of these maps, the family of invariant $K3$ surfaces becomes singular at an invariant 8-cycle \mathcal{R} of rational surfaces (see (7.2)). We show that the restriction $f|_{\mathcal{R}}$ is not birationally conjugate to a surface automorphism: see Appendix C for the maps in Theorem 1 and Proposition 8.2 for the maps in Theorem 3. By Corollary 1.6, then, we have:

Theorem 4. *Let f be a map from Theorems 1 and 3. If $a \neq 1$, then f is not birationally conjugate to an automorphism.*

We note that for birational surface maps, the degree growth of the iterates determines whether the map is birationally conjugate to an automorphism: This occurs if and only if either (i) the degrees are bounded or degree growth is quadratic (see [DiF]), or (ii) if the dynamical degree is a Salem number (see [BC]). Theorem 4 shows that this result does not hold in dimension 3.

We will also determine which mappings $f_{\alpha,\beta}$ are periodic, or finite order, in the sense that $f^p = id$ for some $p > 0$. In contrast to Theorem 4, it was shown by de Fernex and Ein [dFE] that if f is a rational map of finite order, then there is a modification f_X as above, which is an automorphism of X . If f_X is periodic, then f_X^* will also be periodic.

In (4.1) and (4.2) we identify conditions which are necessary for f to be periodic and are sufficient for the existence of a space $Z = Z_{\alpha,\beta}$ such that f_Z is a pseudo-automorphism. We show that for a map in (0.1), if f_Z^* is periodic, then f also turns out to be periodic. The birational map (0.1) may also be considered as a 3-step linear fractional recurrence: given z_0, z_1, z_2 , we define a sequence $\{z_n\}$ by

$$z_{n+3} = \frac{\alpha_0 + \alpha_1 z_n + \alpha_2 z_{n+1} + \alpha_3 z_{n+2}}{\beta_0 + \beta_1 z_n + \beta_2 z_{n+1} + \beta_3 z_{n+2}}. \quad (0.2)$$

The recurrence (0.2) is said to be periodic if the sequence $\{z_n\}$ is periodic for all choices of initial terms z_0, z_1 and z_2 . Equivalently, $f_{\alpha,\beta}^p = id$ for some p . For all $r > 0$ there are r -step recurrences of the form (0.2). In [BK2] we determined the possible periods for 2-step linear

fractional recurrences. McMullen [M1] has explained the periods that arise by showing that the corresponding (2-dimensional) $f_{\alpha,\beta}$ represent certain Coxeter elements.

Here we determine all possible periods for 3-step recurrences (0.2). To rule out trivial cases, we assume that the coefficients satisfy (2.3), and we have:

Theorem 5. *The only nontrivial periods for (0.2) are 8 and 12. Each periodic recurrence is equivalent to one of the following:*

$$z_{n+3} = \frac{1 + z_{n+1} + z_{n+2}}{z_n} \quad z_{n+3} = \frac{-1 - z_{n+1} + z_{n+2}}{z_n} \quad (\text{period } 8)$$

$$z_{n+3} = \frac{\eta/(1-\eta) + \eta z_{n+1} + z_{n+2}}{\eta^2 + z_n} \quad \eta^3 = -1 \quad (\text{period } 12)$$

In the notation of (0.1), the first case corresponds to $\beta = (0, 1, 0, 0)$, $\alpha = (\pm 1, 0, \pm 1, 1)$, and the second case to $\beta = (\eta^2, 1, 0, 0)$, $\alpha = (\eta/(1-\eta), 0, \eta, 1)$.

Each of these mappings has a different structure; these structures are described in Theorems 6.10 and 6.11. The first period 8 recurrence above was found by Lyness [L], and the second one was found by Csörnyei and Laczkovic [CL] (see also [CGM1]). We note that the period 12 recurrences are the case $k = 3$ of a general phenomenon exhibited in [BK4]: For each k , there are k -step linear fractional recurrences with period $4k$. There is a literature dealing with r step recurrences of the form (0.2). We refer to the books [KoL], [KuL], [GL], [CaL] and the extensive bibliographies they contain. That direction of research is largely concerned with the case where the structural parameters α, β , as well as the dynamical points, are real and positive. This avoids the difficulty that the denominator in (0.2) might vanish, causing the expression to be undefined; but the restriction to positive numbers leads to a subdivision into a large number of distinct cases to be treated separately.

In working with the family $f_{\alpha,\beta}$, we work with the pointwise iterates as much as possible, but this runs into difficulties if the orbit enters the indeterminacy locus. We can often deal with this by blowing up certain subsets. In this way we convert these subsets into hypersurfaces, and we then deal with the hypersurfaces by passing to f^* on Pic . This allows us to convert many difficulties with indeterminate orbits into more tractable problems of Linear Algebra.

This paper is organized as follows. §1 assembles some general information about rational maps and the geometry of blowing up. §2 gives the specific behaviors of the maps (0.1). It is evident, then, that there are two possibilities, defined by (3.1), which we call “critical” and “non-critical,” and in §3 we show that any periodic map must be critical. We study the structure of general critical maps in §4. In Theorem 5.1 we show that if f is a critical map satisfying (5.1), then f_Z is a pseudo-automorphism. Pseudo-automorphisms are discussed in §5, together with the possibilities for the induced map f_Z^* on cohomology. In §6 we determine the periodic mappings and give the proof of Theorem 5. In §7 we give the proof of Theorems 1 and 2. At the end of §7 we present a different pseudo-automorphism with positive entropy; it has properties similar to those given in Theorems 1 and 2, but we do not discuss it in detail. The proof of Theorem 3 is given in §8.

§1. Rational Maps A rational map $f : \mathbf{P}^d \dashrightarrow \mathbf{P}^d$ is given by a $(d+1)$ -tuple of homogeneous polynomials, all of the same degree: $f = [f_0 : \cdots : f_d]$. We may divide f by $\text{g.c.d.}(f_0, \dots, f_d)$

so that f_i 's have no common polynomial factor. We define the degree of f , $\deg(f)$, to be the (common) degree of the f_j 's. The *indeterminacy locus* of f is defined by

$$\mathcal{I}(f) = \{x \in \mathbf{P}^d : f_0(x) = \cdots = f_d(x) = 0\}$$

and is a subvariety of codimension at least 2, and f defines a holomorphic mapping $f : \mathbf{P}^d \setminus \mathcal{I}(f) \rightarrow \mathbf{P}^d$. If S is an irreducible subvariety of \mathbf{P}^d , and $S \not\subset \mathcal{I}(f)$, we define the *strict transform*, written simply as $f(S)$, to be the closure of $f(S - \mathcal{I}(f))$. We say that an irreducible variety V is *exceptional* for a rational mapping f if $V \subset \mathcal{I}(f)$, and if the dimension of $f(V - \mathcal{I}(f))$ is strictly less than the dimension of V . Following [DO, p. 64], we say that $f : X \dashrightarrow Y$ is a *pseudo-isomorphism* if f is birational, and if neither f nor f^{-1} has an exceptional hypersurface. It follows that if f is a pseudo-isomorphism, then $f : X \setminus \mathcal{I}(f) \rightarrow Y \setminus \mathcal{I}(f^{-1})$ is biholomorphic. If $X = Y$, we say that f is a *pseudo-automorphism*.

Theorem 1.1. *If $f : X \dashrightarrow Y$ is a pseudo-isomorphism between 3-dimensional manifolds, then the indeterminacy locus has no isolated points.*

Proof. Suppose that there is an isolated point $p \in \mathcal{I}(f)$. Since f^{-1} has no exceptional hypersurfaces, f must blow p up to a curve $C' \subset Y$. Now we consider the behavior of f^{-1} on C' . We must have $C' \subset \mathcal{I}(f^{-1})$, for if f^{-1} is regular at a point $q \in C'$, then f^{-1} must map an open subset of C' to p . Thus the jacobian of f^{-1} must vanish at q . Since the jacobian vanishes on a hypersurface, f^{-1} would have an exceptional hypersurface containing q . Thus q must be indeterminate. Since the total transform of q under f^{-1} is given by $\bigcap_{\epsilon > 0} \overline{(f^{-1}(B(q, \epsilon) - \mathcal{I}(f^{-1})))}$, it must be connected, and it must be a curve C containing p . But since p was an isolated point of $\mathcal{I}(f)$, there are nearby points $p' \in C - \mathcal{I}(f)$. Since f is regular at these points, it must map them to q , and thus f must have an exceptional hypersurface. By this contradiction, we see that $\mathcal{I}(f)$ has no isolated points. \square

For a rational map $f : X \dashrightarrow X$, we consider the iterates $f^j = f \circ \cdots \circ f$, $j > 0$. If Σ is an irreducible hypersurface, then $\Sigma \not\subset \mathcal{I}(f^j)$ for reasons of dimension, so we may consider the sequence of varieties $V_j := f^j(\Sigma)$, for $j > 0$. Since we will be interested in knowing to what extent the iterates of f behave like a pointwise-defined dynamical system, we note: *If $S \not\subset \mathcal{I}(g)$ is irreducible and if $g(S) \not\subset \mathcal{I}(f)$, then $S \not\subset \mathcal{I}(f \circ g)$, and $f(g(S)) = (f \circ g)(S)$.* We may also define f at points of indeterminacy. Let $\gamma_f = \{(x, y) \in (\mathbf{P}^d - \mathcal{I}) \times \mathbf{P}^d : y = f(x)\}$ denote the graph of f at its regular points, and we let Γ denote the closure of γ_f inside $\mathbf{P}^d \times \mathbf{P}^d$. It follows that Γ is an irreducible variety of dimension d , and there are holomorphic projections $\pi_j : \Gamma \rightarrow \mathbf{P}^d$, $j = 1, 2$, onto the first and second factors, respectively, and we have $f = \pi_2 \circ \pi_1^{-1}$ on $\mathbf{P}^d - \mathcal{I}$. For a point $p \in \mathbf{P}^d$, we define the *total transform* to be $f_*p := \pi_2(\pi_1^{-1}p)$, and then we define $f_*(S) := \bigcup_{p \in S} f_*p$. It is easily seen that we have: *If Σ is an irreducible hypersurface, then $f_*(g(\Sigma)) \supset (f \circ g)(\Sigma)$.*

Proposition 1.2. *Suppose that $f : X \dashrightarrow X$ is rational, and suppose that for each exceptional hypersurface E and for $m > 0$, we have $f^m(E - \mathcal{I}) \not\subset \mathcal{I}$. It follows that $(f^*)^n = (f^n)^*$ on $H^{1,1}(X)$.*

Proof. It is sufficient to show that $(f^*)^2 = (f^2)^*$ on $\text{Pic}(X)$. If D is a divisor, then f^*D is the divisor on X which is the same as $f^{-1}D$ on $X - \mathcal{I}$. Since \mathcal{I} has codimension at least 2, we also have $(f^2)^*D = f^*(f^*D)$ on $X - \mathcal{I} - f^{-1}(\mathcal{I})$. By our hypothesis $f^{-1}(\mathcal{I})$ has codimension at least 2. Thus we have $(f^2)^*D = (f^*)^2D$ on X . \square

In a similar way, we may define $f^* : H^{p,q}(X) \rightarrow H^{p,q}(X)$. That is, if β is a (p, q) form on X , then the pullback $\pi_2^* \beta$ is a smooth form on Γ . We may let $[\pi_2^* \beta]$ denote the reinterpretation of the form as a current, and we may push it forward to obtain a current $f^* \beta = \pi_{1*}[\pi_2^* \beta]$ on X . This pulls smooth forms back to currents and is well defined at the level of cohomology classes. If $\alpha \in H^{p',q'}$ is an element of the dual cohomology group, then we have $\langle \alpha, f^* \beta \rangle = \langle \pi_1^* \alpha, \pi_2^* \beta \rangle$. Now if f is birational and $g = f^{-1}$, then

$$\langle g^* \alpha, \beta \rangle = \langle \pi_1^* \alpha, \pi_2^* \beta \rangle = \langle \alpha, f^* \beta \rangle \quad (1.1)$$

If we have $(f^n)^* = (f^*)^n$ on $H^{p,q}$ for $n \geq 0$, then this gives us $(g^n)^* = (g^*)^n$ on $H^{p',q'}$.

The following is proved along standard lines:

Proposition 1.3. *If f is a pseudo-automorphism, then we have $(f^n)^* = (f^*)^n$ on $H^{1,1}$ for all $n \in \mathbf{Z}$. In particular, $(f^{-1})^* = (f^*)^{-1}$.*

From this we get the following:

Proposition 1.4. *If $f : X \dashrightarrow X$ is a pseudo-automorphism on a d -dimensional manifold, then for all $n \in \mathbf{Z}$ we have $(f^*)^n = (f^n)^*$ on both $H^{1,1}$ and $H^{d-1,d-1}$. Further, the characteristic polynomials of f^* on $H^{1,1}$ and $H^{d-1,d-1}$ are the same and therefore the first and the $d-1$ st dynamical degree are the same.*

Proof. From Proposition 1.3, we have $f^*(f^{-1})^* = id$ and so we have that if $\alpha \in H^{1,1}$ and $\beta \in H^{d-1,d-1}$, then $f^* \alpha \cdot f^* \beta = ((f^{-1})^* f^*) \alpha \cdot \beta = \alpha \cdot \beta$. Further, we conclude that $(f^*)^n = (f^n)^*$ on $H^{d-1,d-1}$ for all $n \in \mathbf{Z}$. \square

Now let us define some specific blowup situations. This will serve to define the constructions we will use in the sequel, and it allows us to exhibit the models of indeterminate behavior that we will encounter.

Blowing up a point and a line which contains it. We use $(x_0, x_1, x_2) \mapsto [x_0 : x_1 : x_2 : 1]$ as local coordinates in a neighborhood of $e_3 := [0 : 0 : 0 : 1] \in \mathbf{P}^3$. Let X_1 be the space obtained by blowing up a point e_3 and we let E_3 denote the fiber over e_3 . We may use

$$\pi_1 : X_1 \ni (s_0, s_1, \xi_2)_1 \mapsto [\xi_2 s_0 : \xi_2 s_1 : \xi_2 : 1] \in \mathbf{P}^3 \quad (1.2)$$

as a local coordinate system for a neighborhood of $E_3 \cap \{x_0 = x_1 = 0\}$ in X_1 . It follows that the exceptional fiber $E_3 = \{\xi_2 = 0\}$ in this coordinate system.

Let $\Sigma_{01} = \{x_0 = x_1 = 0\} \subset \mathbf{P}^3$ denote the x_2 -axis. The strict transform of Σ_{01} inside X_1 may be written as $\Sigma_{01} = \{s_0 = s_1 = 0\}$. Thus $\Sigma_{01} \cap E_3 = \{s_0 = s_1 = \xi_2 = 0\}$. Let X_2 be a complex manifold obtained by blowing up Σ_{01} in X_1 . We can define a local coordinate system of X_2 via $\pi_2 : X_2 \ni (t_0, \eta_1, \xi_2)_2 \mapsto (t_0 \eta_1, \eta_1, \xi_2)_1 \in X_1$. Thus $\pi_2 \circ \pi_1 : X_2 \rightarrow \mathbf{P}^3$ is given, in this coordinate neighborhood, by

$$\pi_1 \circ \pi_2 : X_2 \ni (t_0, \eta_1, \xi_2)_2 \mapsto [t_0 \eta_1 \xi_2 : \eta_1 \xi_2 : \xi_2 : 1] \in \mathbf{P}^3. \quad (1.3)$$

The inverse of π_1 (resp. π_2) gives a model of indeterminate behavior that blows up the point $(0, 0, 0)$ (resp. the line $\{x_1 = x_2 = 0\}$) to a hyperplane:

$$\pi_1^{-1} : (x_1, x_2, x_3) \mapsto (x_1/x_3, x_2/x_3, x_3), \quad \pi_2^{-1} : (x_1, x_2, x_3) \mapsto (x_1/x_2, x_2, x_3). \quad (1.4)$$

Blowing up two intersecting lines. Let $\pi_1 : Z_1 \rightarrow \mathbf{P}^3$ be the blowup of the x_1 -axis $\Sigma_{02} = \{x_0 = x_2 = 0\} \subset \mathbf{P}^3$. We use local coordinate system in Z_1

$$\pi_1 : Z_1 \ni (\xi, x, s)_{Z_1} \mapsto [s\xi : x : s : 1] \in \mathbf{P}^3$$

Let us denote the blowup fiber over the point $o = \Sigma_{01} \cap \Sigma_{02} = [0 : 0 : 0 : 1] \in \mathbf{P}^3$ as \mathcal{F}_o^1 then in this coordinate system we have $\mathcal{F}_o^1 = \{s = x = 0\}$. The strict transform of the x_2 -axis in Z_1 is given by $\ell_2 = \{\xi = x = 0\}$ and $\mathcal{F}_o^1 \cap \ell_2 = (0, 0, 0)_{Z_1}$. Now let Z_2 be the blow up of ℓ_2 with a local coordinate system

$$\begin{aligned} \pi := \pi_1 \circ \pi_2 : (t, \eta, s)_{Z_2} \in Z_2 &\mapsto (t, t\eta, s)_{Z_1} \in Z_1 \\ &\mapsto [ts : t\eta : s : 1] \in \mathbf{P}^3 \end{aligned}$$

We denote the second (new) fiber over o as \mathcal{F}_o^2 , so $\mathcal{F}_o^2 = (0, \eta, 0)_{Z_2}$. Let us also use \mathcal{F}_o^1 for its strict transform in Z_2 , so $\mathcal{F}_o^1 \cup \mathcal{F}_o^2 = \pi_2^{-1} \circ \pi_1^{-1} \{o\}$ and $\mathcal{F}_o^1 = (t, 0, 0)_{Z_2}$.

Let $\tau[x_0 : x_1 : x_2 : x_3] = [x_0 : x_2 : x_1 : x_3]$ be the involution that interchanges the x_1 - and x_2 -axes. It follows that τ induces the involution $\tilde{\tau} = \pi^{-1} \circ \tau \circ \pi$ on Z_2 . In coordinates, we have

$$\tilde{\tau} : (t, \eta, s) \mapsto (s/\eta, \eta, t\eta), \quad (1.5)$$

which will serve as our third model of indeterminate behavior. We note that $\tilde{\tau}$ is regular on $\mathcal{F}_o^2 - \mathcal{F}_o^1$, while each point of \mathcal{F}_o^1 blows up to the variety \mathcal{F}_o^1 .

Similarly we can blowup the x_2 -axis first and then the strict transform of x_1 -axis. Performing similar computations, we obtain a blowup space $\hat{\pi} : Y_2 \rightarrow \mathbf{C}^3$. The identity map ι on \mathbf{P}^3 lifts to a map $\tilde{\iota} : Z_2 \rightarrow Y_2$, which in local coordinates is similar to $\tilde{\tau}$.

Remark. Suppose that γ' and γ'' are curves in \mathbf{P}^3 which intersect transversally at points $\{p_1, \dots, p_N\}$. We have local coordinate systems for $1 \leq j \leq N$ so that p_j is the origin, and γ' (resp. γ'') coincides with the x -axis (resp. the y -axis) in a neighborhood of p_j . Since the operation of blowing up the axes is local near p_j , we may construct a blowup space $\pi : W \rightarrow \mathbf{P}^3$ in which γ' and γ'' are both blown up, and over each p_j we are free to choose whether γ' or γ'' was blown up first, independently of the choices over p_k for $k \neq j$.

Theorem 1.5. *Let f be a birational map of X . Let $X_0 \subset X$ be a hypersurface such that the strict transform is $f(X_0) = X_0$. Let $\varphi : X \rightarrow Y$ is a birational map which conjugates (f, X) to an automorphism (g, Y) . Then there is a birational map $\hat{\varphi} : X \rightarrow \hat{Y}$ such that the strict transform $\hat{Y}_0 := \hat{\varphi}(X_0)$ is a nonsingular hypersurface, and the induced map $\hat{g} := \hat{\varphi} \circ f \circ \hat{\varphi}^{-1}$ gives an automorphism of \hat{Y} .*

Proof. We may assume that X_0 is irreducible. Since X_0 is a hypersurface, we may take its strict transform $\varphi(X_0)$. If $\varphi(X_0)$ is a point in Y , then it is fixed by g . If $\pi_1 : Y_1 \rightarrow Y$ be the blowup of the point $\varphi(X_0)$, then g lifts to an automorphism of Y_1 . Let $\phi_1 := \pi_1^{-1} \circ \varphi$. If $\phi_1(X_0)$ is again a point, we can repeat this blowing-up process until $\phi_1(X_0)$ has dimension > 0 , which we may assume to be 1. If the singular locus of $\phi_1(X_0)$ is nonempty, it is finite and invariant under f_1 . Now we can blow up the singular set of $\phi_1(X_0)$ finitely many times and have a new blow up space $\pi_2 : Y_2 \rightarrow Y_1$. Since we were blowing up invariant sets, the induced birational map g_2 of Y_2 is again an automorphism. Now the image $\phi_2(X_0)$ must be a nonsingular curve, which must be invariant. We can blow up this curve, and repeat the process finitely many times so that $\phi_3(X_0)$ has dimension > 1 . We continue this process until $\phi_N(X_0)$ is a nonsingular hypersurface in Y_N , and now we set $\hat{Y} = Y_N$. \square

Corollary 1.6. *Let f be a birational map of X . Let $X_0 \subset X$ be a hypersurface for which the strict transform is $f(X_0) = X_0$. Let $\varphi : X \rightarrow Y$ is a birational map which conjugates (f, X) to an automorphism (g, Y) . Then the restriction (f_{X_0}, X_0) is birationally conjugate to an automorphism.*

Proof of Theorem 4. Let f be as in Theorem 4. In §C we study the restriction of f^8 to the plane $\Sigma_3 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbf{P}^3 : x_3 = 0\}$. There we show that this restricted mapping is not birationally equivalent to an automorphism of Σ_3 . Thus Theorem 4 is a consequence of Corollary 1.6. \square

§2. Linear fractional recurrences. The maps (2.2) are among the Cremona transformations of 3-space which are discussed in Chapter 10 of [H]. We discuss general properties of these transformations, and for the generic parameters (2.10) we construct a new space $\pi : X \rightarrow \mathbf{P}^3$, such that passing to the induced map f_X effectively eliminates one of the exceptional components.

For $\{i_1, \dots, i_k\} \subset \{0, 1, 2, 3\}$, we use the notation

$$\Sigma_{i_1 \dots i_k} = \{x \in \mathbf{P}^3 : x_{i_j} = 0, 1 \leq j \leq k\}, \quad (2.1)$$

and for a vector $A = (A_0, \dots, A_3)$ we will write $A \cdot x = A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3$. In homogeneous coordinates the maps (0.1) take the form

$$f[x_0 : x_1 : x_2 : x_3] = [x_0\beta \cdot x : x_2\beta \cdot x : x_3\beta \cdot x : x_0\alpha \cdot x] \quad (2.2)$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$. In the sequel, we will assume

$$\alpha \neq \lambda\beta, \quad \beta \neq (\beta_0, 0, 0, 0), \quad (\alpha_1, \beta_1) \neq (0, 0). \quad (2.3)$$

Note that if one of the first two conditions does not hold, then f is linear, and if the third condition does not hold, then f is independent of x_1 and thus f is actually a 2-step recurrence. If we set $\gamma = \beta_1\alpha - \alpha_1\beta$, then we have

$$\mathcal{I} = \Sigma_{\beta\gamma} \cup \Sigma_{0\beta} \cup \{e_1\}$$

where $\Sigma_\beta = \{\beta \cdot x = 0\}$, $\Sigma_\gamma = \{\gamma \cdot x = 0\}$, $\Sigma_0 = \{x_0 = 0\}$, $\Sigma_{\beta\gamma} = \Sigma_\beta \cap \Sigma_\gamma$, $\Sigma_{0\beta} = \Sigma_0 \cap \Sigma_\beta$, and $e_1 = [0 : 1 : 0 : 0] = \Sigma_{023}$.

The Jacobian determinant of f is given by $2x_0(\gamma \cdot x)(\beta \cdot x)^2$. Thus we see that the exceptional hypersurfaces are $\mathcal{E} = \{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$. The action of f on the exceptional varieties is given as follows: for $\lambda_2, \lambda_3 \in \mathbf{C}$, $(\lambda_2, \lambda_3) \neq (0, 0)$,

$$\begin{aligned} \Sigma_\beta &\mapsto e_3, \\ f : \quad \Sigma_0 \cap \{\lambda_2x_2 = \lambda_3x_3\} &\mapsto [0 : \lambda_3 : \lambda_2 : 0], \\ \Sigma_\gamma \cap \{\lambda_2x_2 = \lambda_3x_3\} &\mapsto \Sigma_{BC} \cap \{\lambda_2x_1 = \lambda_3x_2\} \end{aligned} \quad (2.4)$$

where we set $\check{\alpha} = (\alpha_0, \alpha_2, \alpha_3, 0)$, $\check{\beta} = (\beta_0, \beta_2, \beta_3, 0)$, and

$$B = (-\alpha_1, 0, 0, \beta_1), \quad C = \beta_1\check{\alpha} - \alpha_1\check{\beta}.$$

Thus Σ_β is blown down to a point. The pencil of lines in Σ_γ passing through $e_1 \in \Sigma_0 \cap \Sigma_\gamma$ are all mapped to points in Σ_{BC} . The pencil of lines in Σ_0 passing through e_1 are all mapped to points on the line Σ_{03} , which is again one of the exceptional lines. We have strict transforms:

$$f : \Sigma_0 \mapsto \Sigma_{03} \mapsto e_1 \quad (2.5)$$

The inverse is given by

$$f^{-1}[x_0 : x_1 : x_2 : x_3] = [x_0 B \cdot x : x_0 \check{\alpha} \cdot x - x_3 \check{\beta} \cdot x : x_1 B \cdot x : x_2 B \cdot x], \quad (2.6)$$

and the indeterminacy locus is $\mathcal{I}(f^{-1}) = \Sigma_{0B} \cup \Sigma_{BC} \cup \{e_3\}$. The Jacobian of f^{-1} is $2x_0(C \cdot x)(B \cdot x)^2$, so the exceptional hypersurfaces are $\mathcal{E}(f^{-1}) = \{\Sigma_0, \Sigma_B, \Sigma_C\}$ and for $\mu_1, \mu_2 \in \mathbf{C}$, $(\mu_1, \mu_2) \neq (0, 0)$,

$$\begin{aligned} \Sigma_B &\mapsto e_1, \\ f^{-1} : \quad \Sigma_0 \cap \{\mu_1 x_1 = \mu_2 x_2\} &\mapsto \Sigma_{0\beta} \cap \{\mu_1 x_2 = \mu_2 x_3\}, \\ \Sigma_C \cap \{\mu_1 x_1 = \mu_2 x_2\} &\mapsto \Sigma_{\beta\gamma} \cap \{\mu_1 x_2 = \mu_2 x_3\}. \end{aligned} \quad (2.7)$$

Now let us construct the space $\pi_1 : X_1 \rightarrow \mathbf{P}^3$ by blowing up a point e_1 , and then the space $\pi_2 : X \rightarrow X_1$ obtained by blowing up a line Σ_{03} . We set

$$\pi = \pi_1 \circ \pi_2 : X \rightarrow \mathbf{P}^3. \quad (2.8)$$

Let S_{03} denote the blowup fiber over the strict transformation of Σ_{03} in X_1 and E_1 for the strict transformation of $\pi_1^{-1}e_1$ in X_1 . For the induced map on X , the orbit of Σ_0 becomes

$$f_X : \Sigma_0 \rightarrow S_{03} \rightarrow E_1 \rightarrow \Sigma_B \quad (2.9)$$

If X and Y are irreducible, we will say that a rational map $f : X \dashrightarrow Y$ is *dominant* if the rank of df is equal to the dimension of Y on a dense open set. Let us define a generic condition:

$$\beta_1 \neq 0, \quad \beta_1 \alpha_2 \neq \alpha_1 \beta_2, \quad \text{and} \quad \beta_1 \alpha_3 \neq \alpha_1 \beta_3. \quad (2.10)$$

For simplicity we use the same notation for both a variety and its strict transform, if there is no possibility of confusion.

Proposition 2.1. *If (2.10) holds, then all the maps in (2.9) are dominant, so $\mathcal{E}(f_X) = \{\Sigma_\beta, \Sigma_\gamma\}$. Further, $\mathcal{I}(f_X) = \Sigma_{\beta 0} \cup \Sigma_{\beta \gamma}$.*

Proof. Let us first consider the restriction to S_{03} . We may use the local coordinates for a neighborhood of S_{03} , $(s_0, x_2, \xi_3)_{S_{03}} \mapsto [s_0 : 1 : x_2 : s_0 \xi_3]$. For the neighborhood of the exceptional fiber E_1 over e_1 , we use $(t_0, \zeta_2, \zeta_3)_{E_1} \mapsto [t_0 : 1 : t_0 \zeta_2 : t_0 \zeta_3]$. It follows that $S_{03} = \{(0, x_2, \xi_3)_{S_{03}}\}$ and $E_1 = \{(0, \zeta_2, \zeta_3)_{E_1}\}$. Using these local coordinates we have

$$f_X|_{S_{03}} : (0, x_2, \xi_3)_{S_{03}} \mapsto \left(0, \xi_3, \frac{\alpha_1 + \alpha_2 x_2}{\beta_1 + \beta_2 x_2}\right)_{E_1}.$$

To have a dominant map, it is required that $\beta_1 \alpha_2 \neq \alpha_1 \beta_2$. For the restrictions of the induced birational map to Σ_0 and E_1 are given by linear maps:

$$\begin{aligned} f_X : \Sigma_0 \ni [0 : x_1 : x_2 : x_3] &\mapsto \left(0, \frac{x_3}{x_2}, \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}\right)_{S_{03}} \in S_{03} \\ f_X : E_1 \ni (0, \zeta_2, \zeta_3)_{E_1} &\mapsto [\beta_1 : \beta_1 \zeta_2 : \beta_1 \zeta_3 : \alpha_1] \in \Sigma_B. \end{aligned}$$

We see that $f_X|_{E_1}$ is dominant because $\beta_1 \neq 0$. And since $\beta_1 \alpha_2 \neq \alpha_1 \beta_2$ and $\beta_1 \alpha_3 \neq \alpha_1 \beta_3$, we see that $f_X|_{\Sigma_0}$ is dominant. \square

Thus in passing to f_X , we have removed one exceptional hypersurface and one point of indeterminacy. There is a group of linear conjugacies acting on the family (0.2). For $(\lambda, c, \mu) \in \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}$, we set

$$(\alpha, \beta) \mapsto (\lambda\alpha, \lambda\beta) \quad (2.11a)$$

$$(\alpha, \beta) \mapsto (\alpha_0, c\alpha_1, c\alpha_2, c\alpha_3, c\beta_0, c^2\beta_1, c^2\beta_2, c^2\beta_3) \quad (2.11b)$$

$$\begin{aligned} (\alpha, \beta) &\mapsto (\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3, \beta'_0, \beta_1, \beta_2, \beta_3) \\ \alpha'_0 &= \alpha_0 + \mu(\alpha_1 + \alpha_2 + \alpha_3) + \mu(\beta_0 + \mu\beta_1 + \mu\beta_2 + \mu\beta_3), \quad \alpha'_1 = \alpha_1 - \mu\beta_1, \\ \alpha'_2 &= \alpha_2 - \mu\beta_2, \quad \alpha'_3 = \alpha_3 - \mu\beta_3, \quad \beta'_0 = \beta_0 + \mu(\beta_1 + \beta_2 + \beta_3). \end{aligned} \quad (2.11c)$$

The first action corresponds to the homogeneity of f . The action (2.11b) corresponds to a scaling of (x_1, x_2, x_3) in affine coordinates, and (2.11c) comes from translation by the vector (μ, μ, μ) . Note that these actions preserve the form of the recurrence relation.

§3. Non-critical maps. A map f of the form (2.2) is *critical* if (3.1) holds:

$$\beta_2 = \beta_3 = 0, \quad \text{and } \beta_1\alpha_2\alpha_3 \neq 0. \quad (3.1)$$

In this section we establish the following:

Theorem 3.1. *If f is not critical, then f is not periodic.*

We will use the following criterion:

Proposition 3.2. *Suppose that $f : X \rightarrow X$ is periodic, i.e., f_X^p is the identity for some $p > 1$. If $E \subset X$ is an exceptional hypersurface, then $f^j E \subset \mathcal{I}(f_X)$ for some $1 \leq j < p$.*

Proof. Since E is exceptional, then $\text{codim}(f(E)) \geq 2$. Let us consider the sequence of varieties $V_j := f_X^j(\Sigma)$. If $V_j \not\subset \mathcal{I}(f_X)$ for all j , then applying the strict transform of f repeatedly, we have $f_X^{j+1}(E) = f_X(V_j)$ for all j , so $\text{codim}(f(V_j)) \geq 2$ for all j . On the other hand, we must have $f_X^p(E) = E = V_p$. \square

The proof of Theorem 3.1 will involve several cases, so we start with some Lemmas.

Lemma 3.3. *Let $\pi : X \rightarrow \mathbf{P}^3$ be the complex manifold defined in (2.8). If $\beta_1 = 0$, then there is a hypersurface $V \subset X$ such that $f_X^n V$ is a point of $X - \mathcal{I}$ either for all $n \geq 1$ or for all $n \leq -1$.*

Proof. We use the local coordinates for a neighborhood of S_{03} and a neighborhood of E_1 defined in §2. Since $\beta \neq (\beta_0, 0, 0, 0)$, we first assume that $\beta_3 \neq 0$ (and thus we may suppose $\beta_3 = 1$) and to consider various cases.

(i) Case $\beta_2 \neq 0$: In this case, the orbit of Σ_β is:

$$f_X : \Sigma_\beta \mapsto e_3 \mapsto (0, 0, \alpha_3)_{S_{03}} \mapsto (0, \alpha_3, \alpha_2/\beta_2)_{E_1} \mapsto e_3 \in \Sigma_0 \setminus \mathcal{I}$$

Thus the orbit is periodic and remains a regular point of f_X .

(ii) Case $\beta_2 = 0, \alpha_2 = 0$. If $\beta_0 + \alpha_3 = 0$, then we have $f_X^2(e_3) = e_3$. If $\beta_0 + \alpha_3 \neq 0$, then $f_X^3(e_3) = e_3$. In both cases, the orbit of Σ_β is pre-periodic.

(iii) Case $\beta_2 = 0, \alpha_2 \neq 0$. With this parameters, we have a two-cycle between Σ_{02} and the fiber over e_2 , $\{(0, 0, \xi_3)_{S_{03}} : \xi_3 \in \mathbf{C}\}$.

$$\begin{aligned} f_X : \quad [0 : x_1 : 0 : x_3] &\in \Sigma_{02} \mapsto (0, 0, (\alpha_1 x_1 + \alpha_3 x_3)/x_3)_{S_{03}} \\ (0, 0, \xi_3)_{S_{03}} &\mapsto [0 : \beta_0 + \xi_3 : 0 : \alpha_2] \in \Sigma_{02}. \end{aligned}$$

Since $f_X^2(e_3) = [0 : \beta_0 + \alpha_3 : 0 : \alpha_2]$ and both Σ_{02} and the fiber over e_2 in S_{03} are disjoint from \mathcal{I} , the forward orbit of Σ_β is disjoint from \mathcal{I} .

(iv) Case $\beta_3 = 0$ and $\beta_2 \neq 0$. Under the backward mapping, Σ_0 is exceptional. To see the backward iteration, let us use a different local coordinate system in a neighborhood of E_1 : $(\zeta_0, \zeta_2, t_3)_{E_1} \mapsto [\zeta_0 t_3 : 1 : \zeta_2 t_3 : t_3] \in \mathbf{P}^3$. Using this local coordinates we see

$$\begin{aligned}\Sigma_0 \ni [0 : x_1 : x_2 : x_3] &\mapsto (0, x_1/x_2, 0)_{E_1} \in E_1 \cap \Sigma_0 \\ f_X^{-1} : E_1 \ni (\zeta_0, \zeta_2, 0)_{E_1} &\mapsto (0, \alpha_1 \zeta_0 / (\beta_2 - \alpha_2 \zeta_0), \zeta_2 / \zeta_0)_{S_{03}} \in S_{03} \\ S_{03} \ni (0, x_2, \xi_3)_{S_{03}} &\mapsto [0 : -\alpha_2 - \alpha_3 x_2 + \beta_2 \xi_3 : \alpha_1 : \alpha_1 x_2] \in \Sigma_0\end{aligned}$$

Let $p = \Sigma_0 \cap E_1 \cap S_{03}$. It follows that $f_X^{-1}p = p$, that is, p is a fixed point for the inverse mapping. \square

Now let us suppose that $\beta_1 \neq 0$. Using actions (2.11a–c), we may assume that $\beta_1 = 1$ and $\alpha_1 = 0$.

Lemma 3.4. *Suppose that $\beta_1 = 1, \alpha_1 = 0$. If either $\beta_2 \neq 0$ or $\beta_3 \neq 0$, then Σ_0 is exceptional and preperiodic for f^{-1} .*

Proof. Let us first assume that $\beta_3 \neq 0$. Then

$$f^{-1} : \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto [0 : -(\beta_2 x_1 + \beta_3 x_2) : x_1 : x_2] \in \Sigma_{0\beta}$$

and $\Sigma_{0\beta}$ is invariant under f^{-1} :

$$f^{-1} : [0 : -(\beta_2 x_2 + \beta_3 x_3) : x_2 : x_3] \mapsto [0 : \beta_2(\beta_2 x_2 + \beta_3 x_3) - \beta_3 x_2 : -(\beta_2 x_2 + \beta_3 x_3) : x_2].$$

Now suppose $\beta_3 = 0$ and $\beta_2 \neq 0$. In this case we have

$$f^{-1} : \Sigma_0 \ni [0 : x_1 : x_2 : x_3] \mapsto [0 : -\beta_2 x_1 : x_1 : x_2] \mapsto [0 : \beta_2^2 : -\beta_2 : 1] \in \Sigma_{0\beta} \setminus \mathcal{I}(f^{-1})$$

and this last point is fixed under f^{-1} . \square

Let $\pi : Z \rightarrow \mathbf{P}^3$ be the complex manifold obtained by blowing up e_2 and Σ_{02} and let E_2 and S_{02} be the corresponding blow-up divisors. In the following Lemma, we use the local coordinates $(s_0, x_1, \xi_2)_{S_{02}} \mapsto [s_0 : x_1 : s_0 \xi_2 : 1]$ in a neighborhood of $S_{02} = \{s_0 = 0\}$ and $(u_0, \eta_1, \eta_3)_{E_2} \mapsto [u_0 : \eta_1 u_0 : 1 : \eta_3 u_0]$ in a neighborhood of $E_2 = \{u_0 = 0\}$.

Lemma 3.5. *Suppose that $\beta_1 = 1, \alpha_1 = \beta_2 = \beta_3 = 0$. If either $\alpha_2 = 0$ or $\alpha_3 = 0$ then S_{02} is exceptional and pre-periodic for f_Z or f_Z^{-1} .*

Proof. If $\alpha_2 = \alpha_3 = 0$ then the mapping is basically linear.

(i) Case $\alpha_3 = 0$ and $\alpha_2 \neq 0$:

$$f_Z : (0, t, \xi_2)_{S_{02}} \mapsto (0, \xi_2, 0)_{E_2} \mapsto (0, \beta_0 + \xi_2, 0)_{S_{02}} \mapsto (0, 0, 0)_{E_2} \leftrightarrow (0, \frac{\beta_0}{\alpha_2}, 0)_{S_{02}}.$$

(ii) Case $\alpha_2 = 0$ and $\alpha_3 \neq 0$:

$$f_Z^{-1} : (0, t, \xi_2)_{S_{02}} \mapsto (0, -\beta_0, \xi_2)_{E_2} \mapsto (0, -\alpha_3, -\beta_0)_{S_{02}} \leftrightarrow (0, -\beta_0, -\beta_0)_{E_2}.$$

\square

Theorem 3.6. *If f is not critical, then there exists a complex manifold X such that either there is an exceptional hypersurface $E \subset X$ for an induced birational map f_X such that $f_X^n E \not\subset \mathcal{I}(f)$ for $n = 1, 2, \dots$, or the analogous statement holds for f_X^{-1} .*

Proof. Let X denote either the space X or Z in the Lemmas above. This Theorem follows from the Lemmas 3.3–5. \square

Proof of Theorem 3.1. If f is not critical, then Theorem 3.6 says that in each case that there is an exceptional hypersurface that does not map into $\mathcal{I}(f_X)$. By Proposition 3.2, then, f is not periodic. \square

§4. Critical Maps. Here we study critical maps in general. Let us recall the condition for f being critical : $\beta_2 = \beta_3 = 0$ and $\beta_1 \alpha_2 \alpha_3 \neq 0$. Using the action (2.11a–c) we may assume that a critical map satisfies:

$$\beta_1 = 1, \beta_2 = \beta_3 = 0, \alpha_1 = 0, \alpha_2 \neq 0, \alpha_3 = 1. \quad (4.1)$$

In this section, we show (Lemma 4.2) that for every critical map there is a blowup space $\pi : Y \rightarrow X$ such that the induced map f_Y has only one exceptional hypersurface, which is Σ_γ . We determine the indeterminacy locus of f_Y (Corollary 4.6) and the dynamical degree for the generic case (Theorem 4.8).

Proposition 4.1. *If f is critical then f^{-1} is conjugate to a critical map.*

Proof. Let $\beta = (\beta_0, 1, 0, 0)$ and $\alpha = (\alpha_0, 0, \alpha_2, 1)$ be parameters of a critical map f . We consider a linear map $\phi : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_3 : x_2 : x_1]$. It follows that we have

$$\begin{aligned} \phi^{-1} \circ f^{-1} \circ \phi : [x_0 : x_1 : x_2 : x_3] \\ \mapsto [x_0 x_1 : x_2 x_1 : x_3 x_1 : x_0(\alpha_0 x_0 - \beta_0 x_1 + x_2 + \alpha_2 x_3)]. \end{aligned}$$

Thus f^{-1} is conjugate to a critical map of the form (2.2) with parameter values $\beta' = (0, 1, 0, 0)$ and $\alpha' = (\alpha_0, -\beta_0, 1, \alpha_2)$ which satisfy the condition (3.1). \square

Remark. By Proposition 4.1, each result for f corresponds to a result for f^{-1} . The translation between f and f^{-1} is guided by notation: $\beta \leftrightarrow B$, $\gamma \leftrightarrow C$, $1 \leftrightarrow 3$: thus $f^{-1} \Sigma_C = \Sigma_{\beta\gamma}$, etc.

If (4.1) holds, it follows that $e_3 = \Sigma_0 \cap \Sigma_\beta \cap \{x_2 = 0\} \in \mathcal{I}$, and

$$f : \Sigma_\beta \dashrightarrow e_3 \rightsquigarrow \Sigma_{01} \rightsquigarrow \Sigma_0 \dashrightarrow \Sigma_{03} \dashrightarrow e_1 \rightsquigarrow \Sigma_B \quad (4.2)$$

We define a new complex manifold $\pi_Y : Y \rightarrow \mathbf{P}^3$ by blowing up e_1 and e_3 , then the strict transform of Σ_{01} , followed by the strict transform of Σ_{03} . (Equivalently, we start with X and blow up the strict transform of e_1 and Σ_{03} .) For $j = 1, 3$, we denote the exceptional divisor over e_j by E_j and the exceptional divisor over Σ_{0j} by S_{0j} for $j = 1, 3$. The induced birational map $f_Y : Y \rightarrow Y$ maps

$$f_Y : \Sigma_\beta \rightarrow E_3 \rightarrow S_{01} \rightarrow \Sigma_0 \rightarrow S_{03} \rightarrow E_1 \rightarrow \Sigma_B \quad (4.3)$$

Lemma 4.2. *The maps in (4.3) are dominant; Σ_γ is the unique exceptional hypersurface for f_Y , and Σ_C is the unique exceptional hypersurface for f_Y^{-1} .*

Proof. Using the local coordinates defined in §2, we have

$$\begin{aligned} \Sigma_\beta \setminus \Sigma_{\beta\gamma} \ni [x_0 : -\beta_0 x_0 : x_2 : x_3] &\mapsto (0, \frac{x_2}{x_0}, \frac{x_3}{x_0})_{E_3} \in E_3 \\ f_Y : E_3 \ni (0, \xi_1, \xi_2)_{E_3} &\mapsto (0, \xi_2, \beta_0 + \xi_1)_{S_{01}} \in S_{01} \\ S_{01} \setminus \Sigma_{\beta\gamma} \ni (0, \eta_1, x_2)_{S_{01}} &\mapsto [0 : x_2(\beta_0 + \eta_1) : (\beta_0 + \eta_1) : 1 + \alpha_2 x_2] \in \Sigma_0. \end{aligned}$$

In Proposition 2.1 we showed that the maps $\Sigma_0 \rightarrow S_{03} \rightarrow E_1 \rightarrow \Sigma_B$ are dominant. It follows that Σ_γ is the only exceptional hypersurface for f_Y , and Σ_C is the only one for f_Y^{-1} . \square

For $p \in \mathbf{P}^3$, we will say that a point of $\pi_Y^{-1}p$ is at *level 1* if it could have been obtained by a blowup a point or curve in \mathbf{P}^3 . Thus the points of all fibers are of level 1, unless they lie over e_1 , e_3 , or $e_2 = \Sigma_{01} \cap \Sigma_{03}$. The fibers $E_1 \cap S_{03}$ and $E_3 \cap S_{01}$ represent the points of E_1 and E_3 which are at level 2. Over e_2 , we define $\mathcal{F}_{e_2}^1 := S_{01} \cap \pi_Y^{-1}e_2$ and $\mathcal{F}_{e_2}^2 := S_{03} \cap \pi_Y^{-1}e_2$. We see that $\mathcal{F}_{e_2}^i$, for $i = 1, 2$ is at level i .

We see that the three curves on level 2 are not indeterminate:

Lemma 4.3. *If f is critical, then the indeterminacy loci $\mathcal{I}(f_Y)$ and $\mathcal{I}(f_Y^{-1})$ do not contain $E_1 \cap S_{03}$, $E_3 \cap S_{01}$, or $\mathcal{F}_{e_2}^2$.*

Proof. Let us first consider the blowup fiber over $E_3 \cap \Sigma_{01}$. For this fiber let us use a local coordinate $(\xi_0, \xi_1, t_2)_{E_3} \mapsto [t_2\xi_0 : t_2\xi_1 : t_2 : 1] \in \mathbf{P}^3$. It follows that the strict transformation of $\Sigma_{01} = \{(0, 0, t_2)_{E_3}\}$ and $E_3 \cap \Sigma_{01} = (0, 0, 0)_{E_3}$. The local coordinates in a neighborhood of the second blowup fiber over $E_3 \cap S_{01}$ is given by $(\eta_0, u_1, t_2)_{E_3^{01}} \mapsto (\eta_0 u_1, u_1, t_2)_{E_3} \mapsto [\eta_0 u_1 t_2 : u_1 t_2 : t_2 : 1] \in \mathbf{P}^3$ and we have the second blowup fiber $E_3 \cap S_{01} = \{(\eta_0, 0, 0)_{E_3^{01}}\}$. With these coordinates, we see that

$$f_Y : (\eta_0, 0, 0)_{E_3^{01}} \mapsto (0, 0, \eta)_{S_{01}} = S_{01} \cap \Sigma_0$$

where $(\xi, t, x_3)_{S_{01}} \mapsto [\xi t : t : 1 : x_3]$ gives a local coordinates near S_{01} . It follows that the second blowup fiber $E_3 \cap S_{01}$ is not indeterminate for f_Y . The computations for f_Y^{-1} and for $E_1 \cap S_{03}$ are essentially the same, and we see that $E_1 \cap S_{03}$ and $E_3 \cap S_{01}$ are not indeterminate for f_Y or to f_Y^{-1} .

To consider the second blowup fiber $\mathcal{F}_{e_2}^2$, we use local coordinates $(\xi, s, x_3)_{01} \mapsto [\xi s : s : 1 : x_3]$. In this coordinates we see that $S_{01} = \{s = 0\}$ and the strict transform of $\Sigma_{03} = \{\xi = 0, x_3 = 0\}$. Thus the local coordinates near the blowup of Σ_{03} is given by $(\eta, s, t)_{03} \mapsto (\eta t, s, t)_{01} \mapsto [\eta t s : s : 1 : t]$ and we have $\mathcal{F}_{e_2}^2 = \{(\eta, 0, 0)_{03}\}$. With this coordinates, we have

$$f_Y : \mathcal{F}_{e_2}^2 \ni (\eta_0, 0, 0)_{03} \mapsto (0, 0, \alpha_2 \eta)_{E_1} = E_1 \cap \Sigma_0$$

where $(\xi_0, t_2, \xi_3)_{E_1} \mapsto [\xi_0 t_2 : 1 : t_2 : \xi_3 t_2]$ is local coordinates near E_1 . Similarly we see that $f_Y^{-1} \mathcal{F}_{e_2}^2 = E_3 \cap \Sigma_0$ and the mapping is dominant. \square

Recall from §2 that in \mathbf{P}^3 each point on $\Sigma_{\beta\gamma}$ blows up to a line in Σ_C . Note that $[0 : 0 : 1 : -\alpha_2] = \Sigma_{\beta\gamma} \cap \Sigma_{01}$, and let $\mathcal{F}_{0\beta\gamma} := \pi_Y^{-1}(\Sigma_{\beta\gamma} \cap \Sigma_{01})$. Note that the base point is the intersection of Σ_{01} and $\Sigma_{\beta\gamma}$, two indeterminate lines. Similarly, we write $\mathcal{F}_{0BC} := \pi_Y^{-1}(\Sigma_{BC} \cap \Sigma_{03})$.

Lemma 4.4. *If f is critical, the fiber $\mathcal{F}_{0\beta\gamma}$ is a component of $\mathcal{I}(f_Y)$, and we have $f_Y : \mathcal{F}_{0\beta\gamma} \setminus \Sigma_{\beta\gamma} \rightsquigarrow \mathcal{F}_{0BC}$.*

Proof. Let us consider a local coordinates in a neighborhood of the fiber $\mathcal{F}_{0\beta\gamma}$ and a local coordinates in a neighborhood of \mathcal{F}_{0BC} :

$$\begin{aligned} (s_0, \eta_1, x)_{S_{01}} &\sim [s_0 : \eta_1 s_0 : 1 : x] \in \mathbf{P}^3 \quad \text{and} \quad \mathcal{F}_{0\beta\gamma} = \{s_0 = 0, x = -\alpha_2\} \\ (s_0, x, \eta_3)_{S_{03}} &\sim [s_0 : 1 : x : \eta_3 s_0] \in \mathbf{P}^3 \quad \text{and} \quad \mathcal{F}_{0BC} = \{s_0 = 0, x = -\alpha_2\}. \end{aligned}$$

Since

$$f_Y(s_0, \eta_1, x)_{S_{01}} = [s_0(\beta_0 + \eta_1) : \beta_0 + \eta_1 : x(\beta_0 + \eta_1) : \alpha_0 s_0 + \alpha_2 + x],$$

we see that for each η_1 we have

$$\begin{aligned} f_Y(0, \eta_1, -\alpha_2)_{S_{01}} &= \lim_{s_0 \rightarrow 0, x \rightarrow -\alpha_2} (s_0, x, \frac{1}{\beta_0 + \eta_1} \frac{\alpha_0 s_0 + \alpha_2 + x}{s_0})_{S_{03}} \\ &= \{(0, -\alpha_2, \eta)_{S_{03}} : \eta \in \mathbf{C}\} = \mathcal{F}_{0BC} \end{aligned}$$

□

Let us define a set $S \subset Y$ to be *totally invariant* if it is completely invariant for the total transform, or if for all $p \in S$, we have $(f_Y)_*p \subset S$ and for all $p \notin S$ we have $(f_Y)_*p \cap S = \emptyset$.

Lemma 4.5. *If f is critical, then Σ_{02} is indeterminate for f_Y . Each point of Σ_{02} blows up to $\mathcal{F}_{e_2}^1$, and $\mathcal{F}_{e_2}^1$ is mapped smoothly to Σ_{02} . The set $\Sigma_{02} \cup \mathcal{F}_{e_2}^1$ is totally invariant.*

Proof. Recall that $f\Sigma_{02} = e_2$, and the point e_2 was blown up. We consider points $[s : 1 : s\xi : x]$ which are close to Σ_{02} when s is small. We see that

$$f_Y : [s : 1 : s\xi : x] \mapsto [\frac{s}{x} : \frac{s\xi}{x} : 1 : s \frac{\alpha_0 s + \alpha_2 s\xi + x}{x(\beta_0 s + 1)}] = (\frac{1}{\xi}, \frac{s\xi}{x}, s \frac{\alpha_0 s + \alpha_2 s\xi + x}{x(\beta_0 s + 1)})_{01}.$$

Letting $s \rightarrow 0$ we see that $f_Y[0 : 1 : 0 : x] \rightsquigarrow \{(\eta, 0, 0)_{01}\}$. Using the same local coordinates we also see that

$$f_Y : \mathcal{F}_{e_2}^1 \ni (\eta, 0, 0)_{01} \mapsto [0 : 1 : 0 : \frac{\alpha_2 \eta}{\beta_0 \eta + 1}] \in \Sigma_{02}.$$

For the second statement, we first notice that from (4.3) $f_Y((\Sigma_{02} \cup \mathcal{F}_{e_2}^1)^c - \mathcal{I}(f_Y))$ is disjoint from the set $\Sigma_{02} \cup \mathcal{F}_{e_2}^1$. Since $\mathcal{I}(f_Y) = \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma} \cup \Sigma_{02}$ and $\Sigma_{\beta\gamma} \cap \Sigma_{02} = \emptyset$, we see that the set $\Sigma_{02} \cup \mathcal{F}_{e_2}^1$ and $\Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma}$ are disjoint. It follows that the set $\Sigma_{02} \cup \mathcal{F}_{e_2}^1$ is totally invariant

□

Corollary 4.6. *If f is critical, then $\mathcal{I}(f_Y) = \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma} \cup \Sigma_{02}$ has pure dimension 1.*

The behavior of f_Y at Σ_{02} is, in suitable coordinates, given by the third model (1.5). The behavior of f_Y at $\mathcal{F}_{0\beta\gamma}$, as seen in Lemma 4.4, is different from the model (1.5). Further, we note that by Proposition 4.1 and the remark following it, the analogues of Lemmas 4.2–5 all hold for f_Y^{-1} . For instance, Σ_C is the unique exceptional hypersurface for f_Y^{-1} , $\mathcal{I}(f_Y^{-1}) = \mathcal{F}_{e_2}^1 \cup \mathcal{F}_{0BC} \cup \Sigma_{BC}$, and each point of $\mathcal{F}_{0BC} - \Sigma_{BC}$ blows up under f_Y^{-1} to $\mathcal{F}_{0\beta\gamma}$.

Corollary 4.7. *If f is critical, then $f_Y^j \Sigma_\gamma \cap (\Sigma_{02} \cup \mathcal{F}_{e_2}^1) = \emptyset$ for all $j \geq 0$.*

Proof. By Lemma 4.5, it suffices to consider the case $j = 0$. By (4.1), $e_2 \notin \Sigma_\gamma$ in \mathbf{P}^3 , so the fiber over e_2 remains disjoint from Σ_γ inside Y . Now $e_1 = \Sigma_{02} \cap \Sigma_\gamma$ in \mathbf{P}^3 and we see that Σ_{02} and Σ_γ are separated when we blow up e_1 to make Y . □

Recall that the degree complexity is $\delta(f) = \lim_{n \rightarrow \infty} (\deg(f^n))^{1/n}$. If $\delta(f) > 1$, then the degrees of the iterates f^n grow exponentially in n . In particular, f cannot be periodic if $\delta(f) > 1$.

Theorem 4.8. *If f is critical, and if $f_Y^n \Sigma_\gamma \not\subset \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma}$, then the first dynamical degree is $\delta(f) \sim 1.32472$, the largest root of $x^3 - x - 1$.*

Proof. Using Corollary 4.7 we see that $f_Y^m \Sigma_\gamma \cap \mathcal{I}(f_Y) = \emptyset$ for all $m \geq 1$. Thus by Proposition 1.2 we have $(f_Y^*)^m = (f_Y^m)^*$ for all m . Thus $\delta(f)$ is the spectral radius of f_Y^* . Inside the Picard group $\text{Pic}(Y)$, we let H_Y be the class of a generic hyperplane in Y , and we have

$$f_Y^* : \begin{aligned} H_Y &\rightarrow 2H_Y - E_1 - E_3 - S_{01}, & S_{01} &\rightarrow E_3 \rightarrow \Sigma_\beta = H_Y - E_3 - S_{01}, \\ E_1 &\rightarrow S_{03} \rightarrow \Sigma_0 = H_Y - E_1 - E_3 - S_{01} - S_{03} \end{aligned} \quad (4.4)$$

The characteristic polynomial of this transformation is $(x^2 + 1)(x^3 - x - 1)$, so $\delta(f)$ is as claimed. \square

Now we give the existence of Green currents, which are invariant currents with the equidistribution properties given in the following:

Theorem 4.9. *If f is as in Theorem 4.8, then there is a positive closed current T_Y^+ in the class of α_Y^+ with the properties: $f_Y^* T_Y^+ = T_Y^+$, and if Ξ^+ is a smooth form which represents α_Y^+ , then $\lim_{n \rightarrow \infty} \delta_1(f)^{-n} f_Y^{n*} \Xi_Y^+ = T_Y^+$ in the weak sense of currents on Y .*

Proof. Recall from Corollary 4.6 that $\mathcal{I}(f_Y) = \Sigma_{02} \cup \Sigma_{\beta\gamma} \cup \mathcal{F}_{0\beta\gamma}$. The total forward image of this set is $\pi_2 \pi_1^{-1} \mathcal{I}(f_Y) = \mathcal{F}_{e_2}^1 \cup \mathcal{F}'_{0BC} \cup \Sigma_C$. We will show that if $\sigma \subset \pi_2 \pi_1^{-1} \mathcal{I}(f_Y)$ is any curve, then $\alpha_Y^+ \cdot \sigma \geq 0$. The Theorem will then be a consequence of Theorem 1.3 of [Ba].

Up to a scalar multiple, we may write $\alpha = H_Y - c_1 E_1 - c_3 E_3 - c_{01} S_{01} - c_{03} S_{03}$. Then since f_Y^* is given by (4.4) we have $1 > c_1 > c_3 > c_{01} = c_{03} > 0$, and $c_1 + c_3 = 1$. Let us start with $\mathcal{F}_{0\beta\gamma} \subset \mathcal{I}(f_Y)$. Points of this curve are blown up to \mathcal{F}_{0BC} . The curve $\sigma = \mathcal{F}_{0BC}$ is the exceptional fiber inside S_{03} over the point $\Sigma_{BC} \cap \Sigma_{03} \in \mathbf{P}^3$. Thus $\sigma \cdot S_{03} = -1$, so $\alpha \cdot \sigma = c_{03} > 0$. Points of the indeterminate curve Σ_{02} blow up to $\sigma = \mathcal{F}_{e_2}^1$. In this case, we have that $\sigma \cdot S_{01}$ and $\sigma \cdot S_{03}$, are ± 1 , with opposite signs, depending on the order of blow-up of Σ_{01} and Σ_{03} . Thus $\sigma \cdot \alpha_Y^+ = \pm c_{01} \mp c_{03} = 0$.

The other possibility is that $\sigma \subset \Sigma_C$. In this case, we have $\sigma \cdot H = \deg(\sigma) = \sigma \cdot S_{01}$. Further, if we let m_3 denote the multiplicity of σ at e_3 , then m_3 is bounded above by $\deg(\sigma)$. If σ is a curve in Σ_C , then it is represented by $L + m_1 \mathcal{F}_{01}^1 + m_3 \epsilon_3$, where \mathcal{F}_{01}^1 represents a fiber of S_{01} , and $\epsilon_3 = E_3 \cap \Sigma_C$. The multiplicities m_1 and m_3 are bounded below by $-\deg(\sigma)$. Since $\mathcal{F}_{01}^1 \cdot S_{01} = -1 = E_3 \cdot \epsilon_3$, we have $\sigma \cdot \alpha_Y^+ \geq \deg(\sigma)(1 - c_{01} - c_3) > 0$. \square

Lemma 4.10. *Suppose that Σ_γ is not preperiodic. If either $f_Y^j \Sigma_\gamma = \mathcal{F}_{0\beta\gamma}$ or $f_Y^j \Sigma_\gamma$ is a point of $\mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}$, then $f_Y^{j+1}|_{\Sigma_\gamma}$ has rank 1, and $f_Y^{j+1} \Sigma_\gamma = \mathcal{F}_{0BC}$.*

Proof. Let us describe the indeterminate behavior of f_Y at $\Sigma_{\beta\gamma}$ and $\mathcal{F}_{0\beta\gamma}$. Up to coordinate changes in domain and range, we may assume that the indeterminate curve is $\{\xi_2 = \xi_3 = 0\}$, and the maps behave like

$$\begin{aligned} (s_1, \xi_2, \xi_3) &\mapsto (s_1, \frac{\xi_2}{\xi_3}, \xi_3) \quad \text{near } \Sigma_{\beta\gamma} \\ (s_1, \xi_2, \xi_3) &\mapsto (s_1 \frac{\xi_2}{\xi_3}, \xi_2, \xi_3) \quad \text{near } \mathcal{F}_{0\beta\gamma} - \Sigma_{\beta\gamma}. \end{aligned}$$

The behavior near $\Sigma_{\beta\gamma}$ is given in (2.7), and $\mathcal{F}_{0\beta\gamma}$ is given in Lemma 4.4.

We will track the forward orbit $f_Y^i \Sigma_\gamma$. Without loss of generality, we may assume that $f_Y^i \Sigma_\gamma$ is not $\Sigma_{\beta\gamma}$ for $1 \leq i \leq j$. Let us choose coordinates (u_1, u_2, v) such that $\Sigma_\gamma = \{v = 0\}$, and the exceptional fibers of Σ_γ are $\{v = 0, u_1 = \text{const}\}$. Thus f has the form $(u_1, u_2, v) \mapsto (u_1, u_2 v, v)$. By making coordinate changes in the range, we may represent the iterated map near Σ_γ as $(u_1, u_2 v, v)$ as long as $f_Y^i \Sigma_\gamma$ is not an exceptional curve in Σ_γ , and not a component of $\mathcal{I}(f_Y)$.

Now suppose that $f_Y^i \Sigma_\gamma = f_Y^{i-1} \Sigma_{BC}$ coincides with $\mathcal{F}_{0\beta\gamma}$, which we will write as $\{\xi_2 = \xi_3 = 0\}$, as above. Thus we may assume that $(s, \xi_2, \xi_3) = (u_1, u_2 v, v)$, so f^{i+1} has the form $(u_1, u_2, v) \mapsto (u_1 u_2, u_2 v, v)$, which is a map of rank 1.

As we continue to iterate f_Y , the other possibility is that $f_Y^i \Sigma_\gamma \subset \Sigma_\gamma$ is an exceptional curve. The coordinate of the map which varies when $v = 0$ must be inside $\{u_1 = \text{const}\} \subset \Sigma_\gamma$, which means that the map must be like $\Sigma_\gamma \ni (u_2 v, u_1 u_2, v) \mapsto (u_2 v, u_1 u_2 v, v)$, which belongs to Σ_{BC} . Now we continue to iterate this point forward. It cannot re-enter Σ_γ , because otherwise the orbit would re-enter Σ_{BC} and become pre-periodic. Thus the only possibility is to enter the indeterminacy locus. Let N denote the first positive integer for which $f_Y^N \Sigma_\gamma \subset \Sigma_\gamma$ was an exceptional curve. Then $\mathcal{I}(f_Y) \cap \bigcup_{i=0}^{N-1} \Sigma_{BC}$ is finite. Thus the forward orbit of the point can intersect the indeterminacy locus only finitely many times. If the point enters $\Sigma_{\beta\gamma} = \{\xi_2 = \xi_3 = 0\}$, then the next image is $(s, \xi_2, \xi_3) = (u_2 v, u_1 u_2 v, v) \mapsto (u_2 v, u_1 u_2, v)$, which has rank 1 again, and we continue as before. \square

§5. Pseudo-automorphisms. In this section we assume that f is critical and we consider the condition

$$f_Y^N \Sigma_\gamma = \Sigma_{\beta\gamma} \text{ for some } N. \quad (5.1)$$

We give conditions for f_Y to be birationally equivalent to a pseudo-automorphism (Theorem 5.1).

Suppose that (5.1) holds. For $1 \leq j \leq N-1$, we consider four possibilities:

- (i) $f_Y^j \Sigma_\gamma$ is a one of the exceptional fibers in Σ_γ , that is $f_Y^j \Sigma_\gamma = \Sigma_\gamma \cap \{\lambda_2 x_2 = \lambda_3 x_3\}$ for some $\lambda_2, \lambda_3 \in \mathbf{C}$ and $f_Y^{j+1} \Sigma_\gamma$ is a point in Σ_{BC} .
- (ii) $f_Y^j \Sigma_\gamma$ is a point of indeterminacy in $\Sigma_{\beta\gamma}$, and this point blows up to one of fibers of Σ_C .
- (iii) $f_Y^j \Sigma_\gamma \subset \mathcal{F}_{0\beta\gamma}$, and $f_Y^{j+1} \Sigma_\gamma = \mathcal{F}_{0BC}$.
- (iv) None of the above.

Theorem 5.1. *Suppose that a critical map f satisfies (5.1), and that whenever case (iii) occurs above, then $f_Y^j \Sigma_\gamma = \mathcal{F}_{0\beta\gamma}$. Then there is a blowup space $\pi : Z \rightarrow Y$ such that f_Z is a pseudo-automorphism.*

Proof. Let us define two sets of subvarieties: $\Lambda_1 = \{f_Y^j \Sigma_\gamma \mid \dim f_Y^j \Sigma_\gamma = 0, 1 \leq j \leq N\}$ and $\Lambda_2 = \{f_Y^j \Sigma_\gamma \mid \dim f_Y^j \Sigma_\gamma = 1, 1 \leq j \leq N\}$. We construct a blowup space $\pi_Z : Z \rightarrow Y$ obtained by first blowing up points in Λ_1 and then blowing up curves in Λ_2 . We denote by $\mathcal{F}_1, \dots, \mathcal{F}_N$ the blowup fibers over $f_Y \Sigma_\gamma = \Sigma_{BC}, f_Y^2 \Sigma_\gamma, \dots, f_Y^N \Sigma_\gamma = \Sigma_{\beta\gamma}$, respectively.

We claim that the induced map f_Z on the orbit of Σ_γ are dominant. To show this we need to check the map f_Z on S_j where $f_Y^j \Sigma_\gamma \subset \mathcal{E}(f_Y)$ or $f_Y^j \Sigma_\gamma \subset \mathcal{I}(f_Y)$. For \mathcal{F}_1 we use local coordinates

$$(x_1, \xi, s)_1 \mapsto [1 : x_1 : -\alpha_0 - \alpha_2 x_1 + s\xi : s] \in \mathbf{P}^3, \quad \text{and } \{s = 0\} = \mathcal{F}_1.$$

Using this we see that

$$f_Z : \Sigma_\gamma \ni [x_0 : x_1 : x_2 : -\alpha_0 x_0 - \alpha_2 x_2] \mapsto \left(\frac{x_2}{x_0}, \beta_0 + \frac{x_1}{x_0}, 0 \right)_1 \in \mathcal{F}_1. \quad (5.2)$$

Suppose the possibility (i) occurs, that is $f_Y^j \Sigma_\gamma = \Sigma_\gamma \cap \{x_2 = \lambda x_3\}$ for some $\lambda \in \mathbf{C}$. Local coordinates near \mathcal{F}_j and \mathcal{F}_{j+1} are given by

$$(s, x, \xi)_j \mapsto [-(\alpha_2 \lambda + 1)/\alpha_0 + s : x : \lambda + s\xi : 1] \in \mathbf{P}^3, \quad \text{and } \{s = 0\} = \mathcal{F}_j$$

$$(s, \xi_1, \xi_2)_{j+1} \mapsto [-(\alpha_2\lambda + 1)/\alpha_0 + s : \lambda + s\xi_1 : 1 : s\xi_2] \in \mathbf{P}^3, \quad \text{and} \quad \{s = 0\} = \mathcal{F}_{j+1}.$$

It follows that

$$f_Z : \mathcal{F}_j \ni (0, x, \xi)_j \mapsto (0, \xi, \frac{(1 + \alpha_2\lambda)(\alpha_0 + \alpha_2\xi)}{\beta_0 - \alpha_0x + \alpha_2\beta_0\lambda})_{j+1} \in \mathcal{F}_{j+1}. \quad (5.3)$$

In case the second possibility (ii), i.e. $f_Y^j \Sigma_\gamma$ is a point in $\Sigma_{\beta\gamma}$ occurs is essentially identical to the first possibility (i). For the third possibility (iii), due to Lemma 6.3 we only need to check the induced map on the blowup fiber over $f_Y^j \Sigma_\gamma = \mathcal{F}_{0\beta\gamma}$. We use local coordinates near \mathcal{F}_j , the blowup fiber over $\mathcal{F}_{0\beta\gamma}$, and \mathcal{F}_{j+1} , the blowup fiber over \mathcal{F}_{0BC} :

$$(s, \eta_1, \eta_2)_j \mapsto (s, \eta_1 + s\eta_2, -1/\alpha_2)_{S_{0,1}} \mapsto [s : s\eta_1 + s^2\eta_2 : -1/\alpha_2 : 1] \in \mathbf{P}^3$$

$$(s, \eta_1, \eta_2)_{j+1} \mapsto (s, -\alpha_2, \eta_1 + s\eta_2)_{S_{0,3}} \mapsto [s : 1 : -\alpha_2 : s\eta_1 + s^2\eta_2] \in \mathbf{P}^3.$$

Using these local coordinates we see that

$$f_Z : \mathcal{F}_j \ni (0, \eta_1, \eta_2)_j \mapsto (0, \frac{\alpha_0}{\beta_0 + \eta_1}, \frac{\alpha_0\eta_2}{\alpha_2(\beta_0 + \eta_1)^2})_{j+1} \in \mathcal{F}_{j+1}. \quad (5.4)$$

Now for the case when $f_Y^j \Sigma_\gamma = E_3 \cap \Sigma_0$, $f_Y^{j+1} \Sigma_\gamma = E_2 \cap \Sigma_0$, $f_Y^{j+2} \Sigma_\gamma = E_1 \cap \Sigma_0$ we use local coordinates

$$(\xi_0, \xi_1, t)_j \mapsto (t\xi_0, \xi_1, t)_{E_3} \mapsto [t^2\xi_0 : t\xi_1 : t : 1] \in \mathbf{P}^3$$

$$(\xi_0, \xi_1, t)_j \mapsto (t\xi_0, \xi_1, t)_{E_2} \mapsto [t^2\xi_0 : t\xi_1 : 1 : t] \in \mathbf{P}^3$$

$$(\xi_0, \xi_1, t)_j \mapsto (t\xi_0, \xi_1, t)_{E_1} \mapsto [t^2\xi_0 : 1 : t\xi_1 : t] \in \mathbf{P}^3.$$

We have

$$f_Z : \mathcal{F}_j \ni (\xi_0, \xi_1, 0)_j \mapsto \left(\frac{\xi_1^2}{\xi_0}, \frac{\xi_1}{\xi_0}, 0 \right)_{j+1} \in \mathcal{F}_{j+1} \quad (5.5)$$

$$f_Z : \mathcal{F}_{j+1} \ni (\xi_0, \xi_1, 0)_j \mapsto \left(\frac{\xi_1^2}{\alpha_2^2\xi_0}, \frac{\xi_1}{\alpha_2\xi_0}, 0 \right)_{j+1} \in \mathcal{F}_{j+1}. \quad (5.6)$$

The last part we have to check is f_Z on \mathcal{F}_N , the blowup fiber of the line of indeterminacy $\Sigma_{\beta\gamma}$. The local coordinates we use near \mathcal{F}_N is given by

$$(s, x_2, \xi_3)_N \mapsto [1 : -\beta_0 + s : x_2 : -\alpha_0 - \alpha_2x_2 + s\xi_3] \in \mathbf{P}^3$$

and we get

$$f_Z : \mathcal{F}_N \ni (0, x_2, \xi_3)_N \mapsto [1 : x_2 : -\alpha_0 - \alpha_2x_2 : \xi_3] \in \Sigma_C. \quad (5.7)$$

From (5.2–7) we see that the induced mapping f_Z is dominant on the orbit of Σ_γ and therefore f_Z has no exceptional hypersurface. By Lemma 4.1, it follows that f_Z^{-1} also has no exceptional hypersurface, so f_Z is a pseudo-automorphism. \square

Lemma 5.2. Suppose that a critical map f satisfies (5.1), and that whenever case (iii) occurs above, then the possibility (iii) can occur at most once.

Proof. Suppose that there are $0 < j_1 < j_2 < N$ such that $f_Y^{j_1} \Sigma_\gamma, f_Y^{j_2} \Sigma_\gamma \subset \mathcal{F}_{0\beta\gamma}$. It follows that $f_Y^{j_1+1} \Sigma_\gamma = f_Y^{j_2+1} \Sigma_\gamma = \mathcal{F}_{0BC}$ and thus \mathcal{F}_{0BC} is fixed under $f_Y^{j_2-j_1}$. Since $0 < j_1 < j_2 < N$, we have that the dimension $f_Y^j \Sigma_\gamma \leq 1$ for all $j = j_1 + 1, \dots, j_2$. Using the fact that \mathcal{F}_{0BC} is fixed under $f_Y^{j_2-j_1}$, we conclude that the dimension $f_Y^j \Sigma_\gamma \leq 1$ for all $j \geq 1$ which contradicts to the assumption that f satisfies (5.1). \square

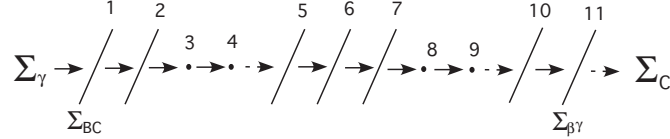


Figure 5.1. A hypothetical orbit: $d_1 = 2, u_1 = 4, d_2 = 7, u_2 = 9, m_d = m_u = 2, N = 11$.

Let N be the smallest positive integer such that $f_Y^N \Sigma_\gamma = \Sigma_{\beta\gamma}$. Let m_d positive integers $d_1 < d_2 < \dots < d_{m_d}$ denote the number of iteration to have the possibility (i), that is for each $j = 1, \dots, m_d$, $f_Y^{d_j} \Sigma_\gamma$ is $\Sigma_\gamma \cap \{\lambda_2 x_2 = \lambda_3 x_3\}$ for some $\lambda_2, \lambda_3 \in \mathbf{C}$. Let $u_1 < u_2 < \dots < u_{m_u}$ be m_u positive integers such that $f_Y^{u_j} \Sigma_\beta$ is of possibility (ii) for $j = 1, \dots, m_u$. We also set m_s be a positive integer such that $f_Y^{m_s+2} \Sigma_\gamma \subset \mathcal{F}_{0\beta\gamma}$ if such possibility occurs. If there is no such case we set $m_s = \infty$. To illustrate this numbering scheme, a hypothetical orbit of Σ_γ is given in Figure 5.1. Here we have assumed that we are in the simpler case $m_s = \infty$, which means that the orbit never enters $\mathcal{F}_{0\beta\gamma}$, so the case (iii) does not occur. Thus in Figure 5.1 the dimension can increase from 0 to 1 only via case (ii).

Let us use the numbers m_s, m_u, m_d, u_j, d_j and N to define four Laurent polynomials:

$$\begin{aligned} Q_1 &:= -1 - \sum_{j=1}^{m_d} \frac{1}{t^{d_j}} + \frac{1}{t^{m_s+1}} \\ Q_2 &:= \frac{1}{t^{m_s}} \left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} \right) \\ Q_3 &:= -1 - \sum_{j=1}^{m_d} \frac{1}{t^{d_j}} + \frac{1}{t^{m_s}} \left(1 + \frac{1}{t} - \frac{1}{t^4} \right) \\ Q_4 &:= -t - t \sum_{j=1}^{m_d} \frac{1}{t^{d_j}} - t \sum_{j=1}^{m_u} \frac{1}{t^{u_j}} - \frac{1}{t^{N-1}} - \frac{1}{t^{m_s}} \left(\frac{1}{t} + \frac{1}{t^3} \right). \end{aligned}$$

Theorem 5.3. If f is pseudo-automorphism and if the possibility (iii) can occur at most once, then the dynamical degree of f is given by the largest root of the polynomial

$$\chi_f(t) := t^{N-1} [(Q_1 - Q_4)t^3 + (2Q_1 - Q_2 - Q_3 - Q_4)t^2 + (Q_1 - Q_3)t + Q_4]. \quad (5.8)$$

Proof. By Corollary 6.2 and Lemma 6.3, we see that f_Y satisfies the hypotheses of Theorem 5.1. Let f_Z be the corresponding pseudo-automorphism. The dynamical degree will be the modulus of the largest root of the characteristic polynomial of f_Z^* . In the Appendix we show that the characteristic polynomial is given by χ_f . \square

§6. Periodic maps. In this section, we determine all possible periodic 3-step recurrences. By §3, we may assume (4.1). The question of periodicities for maps (4.1) with $\beta_0 = 0$ has been answered by Csörnyei and Laczkovic [CL]: they have shown that the only periodicities in this case are the two period 8 maps given in the Theorem stated in the Introduction. We will consider the general case where β_0 is possibly nonzero. We start by giving a necessary condition for a map to be periodic.

Proposition 6.1. *If f is pseudo-automorphism and if E is an exceptional hypersurface then there is an exceptional hypersurface E' for f^{-1} such that $f^n E = E'$ for some $n > 0$ and the co-dimension of $f^j E$ is ≥ 2 for all $j = 1, \dots, n-1$.*

Proof. Suppose f has period p . Since $f^p E = E$ and $\text{codim } fE \geq 2$, it follows that there exists $0 < n \leq p$ such that $\text{codim } f^{n-1} E \geq 2$ and $\text{codim } f^n E = 1$. Thus $f^n E$ is an exceptional for f^{-1} . \square

Since f is critical, $\dim f^j \Sigma_\beta < 2$ for $j = 1, 2$, and $f^3 \Sigma_\beta = \Sigma_0$; further, $\dim f^j \Sigma_0 < 2$ for $j = 1, 2$, and $f^3 \Sigma_0 = \Sigma_B$. By Lemma 4.2 the only exceptional hypersurface for f_Y is Σ_γ , and the only exceptional hypersurface for f_Y^{-1} is Σ_C . This gives us the following necessary condition for f to be periodic.

Corollary 6.2. *If f is periodic, then f is critical and there is some $n > 0$ such that $f_Y^n \Sigma_\gamma = \Sigma_{\beta\gamma}$ and $f_Y^{-n} \Sigma_C = \Sigma_{BC}$.*

Proof. If f is pseudo-automorphism then so is f_Y . Since both f_Y and f_Y^{-1} have the unique exceptional hypersurface, there exists $n \geq 0$ such that $f_Y^n \Sigma_\gamma = \Sigma_{\beta\gamma}$ which blows up to a hypersurface Σ_C . If f is periodic then so is f^{-1} and thus $f_Y^{-n} \Sigma_C = \Sigma_{BC}$. \square

Lemma 6.3. *Suppose f is periodic. If $f_Y^j \Sigma_\gamma \subset \mathcal{F}_{0\beta\gamma}$ (possibility (iii) in §4), then $f_Y^{j+1} \Sigma_\gamma = \mathcal{F}_{0BC}$. Thus f_Z is a pseudo-automorphism.*

Proof. Suppose f is a periodic map with period p . For each $i = 1, \dots, p$, let us set $V_i = f_Y^i \Sigma_\gamma$. It follows that $(f_Y^{-1})^i \Sigma_\gamma = V_{p-i}$. If $V_j \subset \mathcal{F}_{0\beta\gamma}$ then $V_{j+1} = \mathcal{F}_{0BC}$. Applying Lemma 4.10 to f_Y^{-1} we see that $f_Y^{-1} V_{j+1} = V_j = \mathcal{F}_{0\beta\gamma}$. It follows that f satisfies every condition in Theorem 5.1 and therefore f_Z is a pseudo-automorphism. \square

Lemma 6.4. *If f is periodic then $\chi_f(t)$ is self-reciprocal, and $\chi_f = \chi_{f^{-1}}$.*

Proof. A polynomial $p(z) = \sum_{i=0}^k a_i z^i$, $a_i \in \mathbf{C}$ is self-reciprocal if $p(z) = \pm z^k \overline{p(1/\bar{z})}$. If f is periodic then the characteristic polynomial of f_Z^* , $\chi(t)$ is a product of cyclotomic factors and thus $\chi(t)$ is self-reciprocal. Furthermore by Lemma 6.3 f_Z is a pseudo-automorphism and therefore $(f_Z^*)^{-1} = (f_Z^{-1})^*$. It follows that χ_f and $\chi_{f^{-1}}$ are integer polynomials with the same roots. \square

Lemma 6.5. *If f is periodic, then there is a non-negative integer m such that*

- (1) $m = m_u = m_d < N$, $1 < d_1 < u_1 < \dots < d_m < u_m < N$ and
- (2) $N - u_j = d_{m+1-j}$, $N - d_j = u_{m+1-j}$ for $j = 1, \dots, m$.

Proof. From (5.8) we see that the characteristic polynomial $\chi = \chi_f(t)$ is given by $\chi(t) =$

$t^{N-1}(t^2 + 1)\varphi(t)$, where

$$\begin{aligned}\varphi(t) = & \frac{1}{t^{m_s}}(t-1)(t+1+\frac{1}{t}) + \frac{1}{t^{N-1}}(t^N(t^3-t-1) + (t^3+t^2-1)) + \\ & + (t^3+t^2) \left(t \sum_{j=1}^{m_u} \frac{1}{t^{u_j}} + (t-1) \sum_{j=1}^{m_d} \frac{1}{t^{d_j}} \right) - t \sum_{j=1}^{m_u} \frac{1}{t^{u_j}} - t \sum_{j=1}^{m_d} \frac{1}{t^{d_j}}.\end{aligned}\quad (6.1)$$

By Lemma 6.4, $\chi(t)$ should be self-reciprocal. Since the first part of χ and the first line of (6.1) are self-reciprocal, it suffices to consider the case $m_s = \infty$ and $m_u m_d \neq 0$. In this case $\dim f_Z^j \Sigma_\gamma = \dim f_Z^{j+1} \Sigma_\gamma$ if and only if $j \notin \{u_i, i = 1, \dots, m_u\} \cup \{d_i, i = 1, \dots, m_d\}$. Thus it is clear that we have $m = m_u = m_d < N$ and $1 < d_1 < u_1 < \dots < d_m < u_m < N$ for some positive integer m . Thus we have

$$\begin{aligned}f_Z : \Sigma_\gamma \rightarrow \Sigma_{BC} \rightarrow \dots \rightarrow f_Z^{d_1} \Sigma_\gamma \rightarrow p_1 \in \Sigma_{BC} \rightarrow \dots \rightarrow q_1 \in \Sigma_{\beta\gamma} \\ \rightsquigarrow f_Z^{u_1+1} \Sigma_\gamma \subset \Sigma_C \rightarrow \dots \rightarrow f_Z^{d_2} \Sigma_\gamma \rightarrow \dots \rightarrow f_Z^N \Sigma_\gamma = \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C\end{aligned}$$

By interchanging the roles of Σ_β , Σ_γ and Σ_B , Σ_C , we see that the characteristic polynomial for f^{-1} is given by $\hat{\chi}_{f^{-1}}(t) = t^{N-1}(t^2 + 1)\hat{\varphi}(t)$ where

$$\begin{aligned}\hat{\varphi}(t) = & \frac{1}{t^{N-1}}(t^N(t^3-t-1) + (t^3+t^2-1)) + \\ & + (t^3+t^2) \left(t \sum_{j=1}^m \frac{1}{t^{N-d_j}} + (t-1) \sum_{j=1}^m \frac{1}{t^{N-u_j}} \right) - t \sum_{j=1}^m \frac{1}{t^{N-d_j}} - t \sum_{j=1}^m \frac{1}{t^{N-u_j}}.\end{aligned}\quad (6.2)$$

Since both f and f^{-1} have the same characteristic polynomial, we obtain the second statement of the Lemma by comparing χ_f and $\chi_{f^{-1}}$. \square

Lemma 6.6. *Suppose f is periodic.*

- (a) *If m is even, then for all $j = 1, \dots, m$, $2 \leq u_j - d_j \leq d_1$.*
- (b) *If m is odd, then $1 \leq u_{(m+1)/2} - d_{(m+1)/2} \leq d_1$ and for all $j \neq (m+1)/2$, $2 \leq u_j - d_j \leq d_1$.*

Proof. Suppose j_* is the smallest positive integer such that $u_{j_*} - d_{j_*} > d_1$. Then we have (1) $f_Y^{d_{j_*}} \Sigma_\gamma$ is a fibration in Σ_γ , (2) $f_Y^{d_{j_*}+i} \Sigma_\gamma \in$ a point in $f_Y^i \Sigma_\gamma$ for $i = 1, \dots, d_1$, and (3) $f_Y^{d_{j_*}+d_1+1} \Sigma_\gamma \in f_Y^{d_1+1} \Sigma_\gamma$ which is a point in Σ_{BC} . It follows that the exceptional hypersurface Σ_γ is pre-periodic which contradicts to the hypothesis f is periodic. If $u_j - d_j = 1$ then $f_Y^{d_j+1} \Sigma_\gamma = \Sigma_{BC} \cap \Sigma_{\beta\gamma} = f_Y^{u_j} \Sigma_\gamma$. Thus the situation $u_j - d_j = 1$ can happens at most once and by Lemma 6.5 we see that $(N - d_j) - (N - u_j) = u_{m-j+1} - d_{m-j+1} = 1$. It follows that $j = m - j + 1$ and thus $j = (m+1)/2$. \square

Lemma 6.7. *Suppose f is critical and $m \geq 1$ then*

- (a) $d_1 \neq 1, 3, 4$
- (b) *If $d_1 = 2$ then $m = 1$, and either (i) $\alpha_0 = \alpha_2 = 1$ and $\beta_0 = 0$ or (ii) $\alpha_0 = \eta^2 \alpha_2 = \eta, \beta = \eta^2$ where $\eta^2 - \eta + 1 = 0$.*
- (c) *If $m \geq 2$ is odd then for $j = 1, \dots, m-1$, $d_{j+1} - u_j \geq 5$.*

- (d) If $m \geq 2$ is even then for $1 \leq j \leq m-1$ and $j \neq m/2$, $d_{j+1} - u_j \geq 5$ and $d_{m/2+1} - u_{m/2} \geq 4$.

Proof. (a) $d_1 = 1$ means Σ_{BC} is a line through e_1 in Σ_γ . Since $\Sigma_{BC} = \{x_3 = 0, \alpha_0 x_0 + \alpha_2 x_1 + x_2 = 0\}$ and $\alpha_2 \neq 0$, it follows that $e_1 \notin \Sigma_{BC}$ and thus $d_1 \neq 1$. Since $\Sigma_{BC} \subset \Sigma_3$, we have $f_Y^3 \Sigma_\gamma = f_Y^2 \Sigma_{BC} \subset \Sigma_1$ which doesn't contain e_1 . Furthermore $f_Y \Sigma_1 = \{[\beta_0 x_0 : \beta_0 x_2 : \beta_0 x_3 : \alpha_0 x_0 + \alpha_2 x_2 + x_3] : \beta_0 \neq 0\}$ if $\beta_0 \neq 0$ and $f_Y \Sigma_1$ is a line in the blowup fiber E_3 if $\beta_0 = 0$. It follows that $f_Y^4 \Sigma_\gamma$ does not contain e_1 . Therefore $d_1 \neq 3$ or 4 . The statement for (b) can be confirmed by direct computation. For each $j \leq m-1$, $f_Y^{u_j+1} \Sigma_\gamma$ is a line through e_3 in Σ_C which can be parametrized as $t \mapsto \{[1 : \mu : -\alpha_0 - \alpha_2 \mu : t]\}$ for some fixed $\mu \in \mathbf{C} \cup \{\infty\}$. By computing the forward iteration of $[1 : \mu : -\alpha_0 - \alpha_2 \mu : t]$ we see that $d_{j+1} - u_j \neq 1, 2$, or 3 . Furthermore $d_{j+1} - u_j = 4$ if and only if $f_Y^{u_j+1} \Sigma_\gamma = \Sigma_C \cap \{(1 + \alpha_2 \beta_0)x_0 + \alpha_2 x_1 = 0\}$. It follows that $d_{j+1} - u_j = 4$ occurs only once. Suppose $d_{j+1} - u_j = 4$ for some $1 \leq j \leq m-1$. By Lemma 6.5 we see that $(N - u_j) - (N - d_{j+1}) = d_{m-j+1} - u_{m-j} = 4$. It follows that $j+1 = m-j+1$ and thus $j = m/2$. The statement (c) and (d) follows. \square

Direct computation shows the following properties:

Lemma 6.8. Suppose f is critical, then $\hat{\varphi}(t)$ defined in (6.2) satisfies

- (a) $\hat{\varphi}(1) = 0$, and
- (b) $\hat{\varphi}'(1) = 7(m+1) - (N + \sum_{j=1}^m (u_j - d_j))$.

Lemma 6.9. Suppose that $m \geq 2$ and that f is critical satisfying (5.1). Then

$$\begin{aligned} N + \sum_{j=1}^m (u_j - d_j) &\geq 9m + 3 && \text{if } m \text{ is odd} \\ &\geq 9m + 4 && \text{if } m \text{ is even} \end{aligned}$$

Thus $\hat{\varphi}'(1) < 0$, so $\hat{\varphi}$ has a root greater than 1.

Proof. Suppose m is even. By Lemma 6.5 we see that

$$\begin{aligned} N + \sum_{j=1}^m (u_j - d_j) &= 2d_1 + 4(u_1 - d_1) + 2(d_2 - u_1) + \cdots \\ &\quad \cdots + 2(d_{m/2} - u_{m/2-1}) + 4(u_{m/2} - d_{m/2}) + (d_{m/2+1} - u_{m/2}) \end{aligned}$$

By Lemma 6.7, (a) and (b), we have $d_1 \geq 5$. Applying Lemma 6.6 (a) and Lemma 6.7 (d) we have

$$N + \sum_{j=1}^m (u_j - d_j) \geq 2 \cdot 5 + 4 \cdot 2 + 2 \cdot 5 + \cdots + 4 \cdot 2 + 4 = 9m + 4.$$

Similarly when m is odd

$$\begin{aligned} N + \sum_{j=1}^m (u_j - d_j) &= 2d_1 + 4(u_1 - d_1) + 2(d_2 - u_1) + \cdots \\ &\quad \cdots + 2(d_{(m+1)/2} - u_{(m-1)/2}) + 2(u_{(m+1)/2} - d_{(m+1)/2}). \end{aligned}$$

Again applying Lemma 6.6 (b) and Lemma 6.7(c) we have

$$N + \sum_{j=1}^m (u_j - d_j) \geq 2 \cdot 5 + 4 \cdot 2 + 2 \cdot 5 + \cdots + 2 \cdot 5 + 2 = 9m + 3.$$

\square

Theorem 6.10. *If f is periodic with $m = 0$ and $m_s = \infty$ then f is one of the following:*

- $\alpha = (-1, 0, -1, 1)$, $\beta = (0, 1, 0, 0)$: $f_{\alpha\beta}$ has period 8 and there is a conic Q such that

$$f_Y : \Sigma_\gamma \rightarrow \Sigma_{BC} \rightarrow Q \rightarrow \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C.$$

- $\alpha = (-1/2, 0, -1, 1)$, $\beta = (1, 1, 0, 0)$: $f_{\alpha\beta}$ has period 12, and

$$f_Y : \Sigma_\gamma \rightarrow \Sigma_{BC} \rightarrow L_1 \rightarrow L_2 \rightarrow \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C$$

where we set $L_1 = \Sigma_2 \cap \{x_0 + x_3 = 0\}$ and $L_2 = \Sigma_1 \cap \{x_0 + x_2 = 0\}$.

Proof. The polynomial defined in (5.8) is also given by

$$\chi(t) = (t^2 + 1) (t^N (t^3 - t - 1) + t^3 + t^2 - 1).$$

It follows that $\chi(t)$ has a root bigger than 1 if and only if $N \geq 8$ and in case $N = 7$ the matrix representation of f_Z^* has 3×3 Jordan block with eigenvalue 1. Thus we need to check the situation $f^{n+1}\Sigma_\gamma = \Sigma_{\beta\gamma}$ only for $n \leq 5$. For this, let us parametrize $\Sigma_{BC} = \{[1 : t : -\alpha_0 - \alpha_2 t : 0]\}$ and let $[f_0^{(n)} : f_1^{(n)} : f_2^{(n)} : f_3^{(n)}]$ denote the n -th iteration of Σ_{BC} . If $f^{n+1}\Sigma_\gamma = \Sigma_{\beta\gamma}$ then for all t we have

$$\beta_0 f_0^{(n)} + f_1^{(n)} = 0, \quad \text{and} \quad \alpha_0 f_0^{(n)} + \alpha_2 f_2^{(n)} + f_3^{(n)} = 0 \quad (6.3)$$

Since equations in (6.3) are polynomials in t whose coefficients are integer polynomials in the variables β_0, α_0 , and α_2 , we may use the computer show that for $0 \leq n \leq 5$, the only two possibilities are those listed above. \square

Theorem 6.11. *If f is periodic with $m = 1$ and $m_s = \infty$, then f is one of the following:*

- $\alpha = (1, 0, 1, 1)$, $\beta = (0, 1, 0, 0)$: f has period 8, $\Sigma_{BC} \cap \Sigma_{\beta\gamma} \neq \emptyset$, and

$$f_Y : \Sigma_\gamma \rightarrow \Sigma_{BC} \rightarrow \Sigma_\gamma \cap \Sigma_2 \rightarrow \Sigma_{BC} \cap \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C \cap \Sigma_2 \rightarrow \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C.$$

- $\alpha = (\eta/(1 - \eta), 0, \eta, 1)$, $\beta = (\eta^2, 1, 0, 0)$ and $\eta^3 = -1$, $\eta \neq -1$: $f_{\alpha\beta}$ has period 12, and

$$f_Y : \Sigma_\gamma \rightarrow \Sigma_{BC} \rightarrow \Sigma_\gamma \cap \Sigma_2 \rightarrow p_1 \in \Sigma_{BC} \rightarrow p_2 \in \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C \cap \Sigma_1 \rightarrow \Sigma_{\beta\gamma} \rightsquigarrow \Sigma_C,$$

where $p_1 = [1 : 0 : -\eta^2 : 0] \in \Sigma_{BC}$ and $p_2 = [1 : -\eta^2 : 0 : -\eta^2] \in \Sigma_{\beta\gamma}$.

Proof. From (5.8) the characteristic polynomial of f_Z^* is given by

$$\chi(t) = t^{N-(u_1+d_1)}(t^2 + 1)(t^{d_1} + 1) (t^{u_1}(t^3 - t - 1) + t^3 + t^2 - 1).$$

It follows that $\chi(t)$ has a root bigger than 1 if and only if $u_1 \geq 8$. If $u_1 = 7$, the f_Z^* has a 3×3 Jordan block. Thus If f_Z^* is periodic then $d_1 \leq 5 < u_1$. By direct computation of $f^n \Sigma_\gamma = f^{n-1} \Sigma_{BC}$ for $n = 1, \dots, 5$, we can easily check the two conditions (i) $f^{n-1} \Sigma_{BC} \subset \Sigma_\gamma$, (ii) $f^{n-1} \Sigma_{BC} \subset \{x_3 = \lambda x_2\}$ for some $\lambda \in \mathbf{C}$ and thus we see that there are only two possibilities listed in this Theorem. \square

Theorem 6.12. *If $m \geq 2, m_s = \infty$ then f has exponential degree growth (and is not periodic).*

Proof. By Lemmas 6.8 and 6.9 we see that $\chi_N(1) = 0$ and $\chi'_N(1) = 2\dot{\varphi}'(1) < 0$. Since the leading coefficient of χ_N is 1, there exist a real root which is strictly bigger than 1. It follows that the dynamical degree of f is strictly bigger than 1. \square

Theorem 6.13. *If $1 \leq m_s < \infty$, then f is not periodic.*

Proof. By Lemmas 6.8 and 6.9 we see that $\chi_N(1) = 0$ and $\chi'_N(1) = 2(3 + \dot{\varphi}'(1)) = 2(-2m + 1)$. It follows that if $m \geq 1$ then f has positive entropy. Now suppose $m = 0$, we have $\chi_N(t) = (t^2 + 1)(t^N(t^3 - t - 1) + t^{N-m_s-2}(t-1)(t^2 + t + 1) + t^3 + t^2 - 1)$. If f is periodic then the characteristic polynomial for f_Z^* should be self-reciprocal. It follows that $m_s = (N - 4)/2$. It follows that N should be even and

$$\chi_N(t) = (t^2 + 1) \left(t^N(t^3 - t - 1) + t^{N/2}(t-1)(t^2 + t + 1) + t^3 + t^2 - 1 \right).$$

By inspection we see that $m_s \geq 3$ and it follows that $N \geq 10$. We can also check that $\chi_N(1) = 0$ and $\chi'_N(t) = 10 - N$. Therefore if $m_s \neq 3$ then f is not periodic. In case $m_s = 3$, the matrix representation for f_Z^* has 3×3 Jordan block with eigenvalue 1 and all other eigenvalues have modulus 1. \square

Proof of Theorem 5: The statement of the Theorem 5 in the Introduction follows from Theorems 6.10–13. \square

We remark that in the proof of Theorem 6.13, we see that if $m_s = 3$ and $m = 0$, then the degree of f^n is quadratic in n . This case occurs for $\alpha = (a, 0, 1, 1)$ and $\beta = (0, 1, 0, 0)$, which is the so-called Lyness process and will be discussed in §8.

§7. Pseudo-automorphisms with positive entropy In this section we consider the case

$$\beta = (0, 1, 0, 0) \quad \text{and} \quad \alpha = (a, 0, \omega, 1) \quad (7.1)$$

where $\omega^2 + \omega + 1 = 0$ and $a \in \mathbf{C} \setminus \{0\}$. With this choice of parameters, we see that f is critical and that $\Sigma_B = \Sigma_3$ and $\Sigma_\beta = \Sigma_1$. Since the maps $f : \Sigma_3 \rightarrow \Sigma_2 \rightarrow \Sigma_1$ are dominant, (4.3) gives an 8-cycle of dominant maps

$$f_Y : \Sigma_1 \rightarrow E_3 \rightarrow S_{01} \rightarrow \Sigma_0 \rightarrow S_{03} \rightarrow E_1 \rightarrow \Sigma_3 \rightarrow \Sigma_2 \rightarrow \Sigma_1 \quad (7.2)$$

Since this 8-cycle is fundamental to our understanding of f in this case, we will refer to the union of these 8 hypersurfaces as the *rotor* and denote it as \mathcal{R} . Clearly, f_Y^8 fixes each component of the rotor; in addition, it has a relatively simple expression. On Σ_3 , for example, we have:

$$\begin{aligned} f_Y^8 : \Sigma_3 \ni [x_0 : x_1 : x_2 : 0] \\ \mapsto [x_0(ax_0 + \omega x_2)(ax_0 + ax_1 + \omega x_2) \\ : x_1(x_1 x_2 + a\omega x_0^2 + a\omega x_0 x_1 + a\omega x_0 x_2 + \omega^2 x_0 x_2 + \omega^2 x_2^2) \\ : \omega x_2(ax_0 + \omega x_2)(x_1 + a\omega x_0 + \omega^2 x_2) : 0] \in \Sigma_3. \end{aligned} \quad (7.3)$$

The restriction of f_Y^8 to the rotor is studied in §C.

Note that by (7.1), $\Sigma_{BC} = \Sigma_3 \cap \Sigma_C$ and $\Sigma_{\beta\gamma} = \Sigma_1 \cap \Sigma_\gamma$. Using (7.2) we may verify that f_Y satisfies condition (5.1), which in this case is

$$f_Y^j \Sigma_\gamma \not\subset \mathcal{F}_{0\beta\gamma} \text{ for all } 1 \leq j \leq 10, \text{ and } f_Y^{11} = \Sigma_{\beta\gamma} \quad (7.4)$$

We define the space $\pi_Z : Z \rightarrow Y$ by successively blowing up the 11 curves $\gamma_j := f_Y^j \Sigma_\gamma$, $1 \leq j \leq 11$. The dynamical degree, being a birational invariant, is independent of the order in which the γ_j 's are blown up.

Theorem 7.1. *The induced map f_Z is a pseudo-automorphism, and the dynamical degree of f is greater than 1.*

Proof. From (7.4) we see that f_Y satisfies the condition in Theorem 5.1, so f_Z is a pseudo-automorphism. By Theorem 5.3, the characteristic polynomial of f_Z^* is $t^{11}(t^3 - t - 1) + t^3 + t^2 - 1 = (-1 + t)(1 + t)(1 + t^4)(1 - t^3 - t^4 - t^5 + t^8)$. Thus $\delta(f)$ is the largest root of this polynomial, which is approximately 1.28064. \square

The space Z has been defined earlier, but now let us be more precise: we define Z as the space obtained by blowing up first $\gamma_{11} \subset Y$, then we blow up the strict transform of γ_{10} in the resulting space, followed by blowing up the strict transform of γ_9 , and continuing this way until we blow up the strict transform of γ_1 . We will use the notation Γ_j to denote the exceptional divisor of the blowup of γ_j . There are no points where three distinct γ_j 's intersect. If $p = \gamma_j \cap \gamma_k$, with $j > k$, then we blow up γ_j first, and we refer to the fiber in Γ_j over p as the *first* fiber over p , and write it as \mathcal{F}_p^1 . We then blow up the strict transform of γ_k , and the blowup fiber over the point $\gamma_k \cap \Gamma_j$ is equal to $\Gamma_j \cap \Gamma_k$.

Let us describe some of the intersections of the γ_j 's. f is constant on each line in Σ_γ passing through e_1 . Further, $\gamma_1 \subset \Sigma_3$, $\gamma_2 \subset \Sigma_2$, and $\gamma_6 \subset \Sigma_0$, and $e_1 = \Sigma_0 \cap \Sigma_2 \cap \Sigma_3$. We set $\ell_2 = \Sigma_\gamma \cap \Sigma_3$, $\ell_3 = \Sigma_\gamma \cap \Sigma_2$, and $\ell_7 = \Sigma_\gamma \cap \Sigma_0$. Thus we have $f(\ell_j) = \gamma_1 \cap \gamma_j$ for $j = 2, 3, 7$. The curve $\gamma_9 \subset \Sigma_1$ is a conic, and $\gamma_9 \cap \gamma_1$ consists of two points. We let ℓ'_9 and ℓ''_9 denote the two lines in Σ_γ for which $f(\ell'_9 \cup \ell''_9) = \gamma_1 \cap \gamma_9$. This accounts for all the curves γ_j which intersect γ_1 . As a consequence of the order of blowup, the first fiber $\mathcal{F}_{f(\ell_j)}^1 = \mathcal{F}_{\gamma_1 \cap \gamma_j}^1$, $j = 2, 3, 7$ is contained in Γ_j and similarly for ℓ'_9, ℓ''_9 .

There is a similar situation for the γ_j 's which intersect γ_{11} . The curves γ_5, γ_9 and γ_{10} each intersect γ_{11} in a single point, and γ_3 , which intersects γ_{11} in 2 points, and this accounts for all the intersection points between γ_{11} and the other γ_j 's.

Let us use the notation $\pi_1 : Z_1 \rightarrow Y$ for the manifold obtained by blowing up the curve $\gamma_{11} \subset Y$. This is the first blowup performed in the construction of Z . Let $f_{Z_1} : Z_1 \dashrightarrow Z_1$ be the induced map. Since $\mathcal{I}(f_Y) = \gamma_{11} \cup \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}^1$, it follows that $\mathcal{I}(f_{Z_1}) \subset \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}^1 \cup \gamma_{10} \cup \Gamma_{11}$.

Lemma 7.2. $\mathcal{I}(f_{Z_1}) = \mathcal{F}_{0\beta\gamma}^1 \cup \Sigma_{02} \cup \gamma_{10}$.

Proof. We have seen already that the indeterminacy locus is contained in $\Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}^1 \cup \gamma_{10} \cup \Gamma_{11}$, so it suffices to show that $\mathcal{I}(f_{Z_1}) \cap \Gamma_{11}$ consists of the two points $\gamma_{10} \cap \Gamma_{11}$ and $\mathcal{F}_{0\beta\gamma}^1 \cap \Gamma_{11}$. Thus we look at f_{Z_1} in coordinate charts that cover Γ_{11} . We will look first at $\Gamma_{11} \cap \pi_1^{-1}(\gamma_{11} - \gamma_{10})$.

In the local coordinates $(s, \zeta, x_3)_{S_{01}} \mapsto [s : s\zeta : 1 : x_3] \in \mathbf{P}^3$ in the neighborhood of $S_{01} - E_1 = \{s = 0, \zeta \neq \infty\}$, we have $\gamma_{11} = \{\zeta = 0, as + \omega + x_3 = 0\}$ and $\mathcal{F}_{0\beta\gamma}^1 \cap \gamma_{11} = (0, 0, -\omega)_{S_{01}}$. We use the local coordinate charts $(s, t, \eta)'$ on U' and $(s, \eta, t)''$ on U'' so that π_1 is given by

$$\begin{aligned} \pi' : U' \ni (s, t, \eta)' &\mapsto (s, t, -as - \omega + t\eta)_{S_{01}} \\ \pi'' : U'' \ni (s, \eta, t)'' &\mapsto (s, t\eta, -as - \omega + t)_{S_{01}} \end{aligned}$$

It is evident that $\Gamma_{11} \supset \{t = 0\}$ in both coordinate charts, and $U' \cup U'' \supset \pi_1^{-1}(\gamma_{11} - \gamma_{10})$. The induced map $f \circ \pi_1 : U' \cup U'' \rightarrow \mathbf{P}^3$ is given by

$$\begin{aligned} U' \ni (s, t, \eta)' &\mapsto [s : 1 : -as + t\eta - \omega : \eta] \\ U'' \ni (s, \eta, t)'' &\mapsto [s\eta : \eta : \eta(-as + t - \omega) : 1]. \end{aligned} \tag{7.4}$$

so we see that $\{t = 0\}$ is mapped to Σ_C .

From (7.4) we see that the map $f \circ \pi_1 : U' \cup U'' \rightarrow \mathbf{P}^3$ is everywhere regular. The only points of Σ_C which is blown up in the construction of Y are e_3 and $[0 : 1 : -\omega : 0]$ which is point the base point of \mathcal{F}_{0BC}^1 . By (7.4), the preimage of e_3 is $(0, 0, 0)'' \in U''$, and the preimage $[0 : 1 : -\omega : 0]$ is $(0, 0, 0)' \in U'$. Working in local coordinates in Y over e_3 , we find that $f \circ \pi_1 : U'' \rightarrow Y$ is everywhere regular. Thus we conclude that $f_Y \circ \pi_1 : Z_1 \rightarrow Y$ is regular on $(U' - (0, 0, 0)') \cup U''$. Now in order to pass to f_{Z_1} we need to consider the point $\gamma_{11} \cap \Sigma_C$ which is blown up. However, this is the image point of $\gamma_{10} \cap \gamma_{11}$, which is not in our coordinate chart. We note that $(0, 0, 0)'$ is the point $\mathcal{F}_{0\beta\gamma}^1 \cap \Gamma_{11}$, so we conclude that f_{Z_1} is regular at all points of $\Gamma_{11} - (\pi_1^{-1}(\gamma_{11} \cap \gamma_{10}) \cup \mathcal{F}_{0\beta\gamma}^1)$.

Now we consider $\Gamma_{11} \cap \pi^{-1}(\gamma_{11} - \mathcal{F}_{0\beta\gamma}^1)$, which does not lie over any of the centers of blowup in the construction of Y . As in (5.7), we may use the local coordinates $(s, x_2, \zeta) \mapsto [1 : s : x_2 : -a - \omega x_2 + s\zeta]$ in a neighborhood of $\{s = 0, \zeta \neq \infty\} \subset \Gamma_{11} - \pi_1^{-1}(\mathcal{F}_{0\beta\gamma}^1)$, and we get

$$f_{Z_1} : \Gamma_{11} \ni (0, x_2, \zeta) \mapsto [1 : x_2 : -a - \omega x_2 : \zeta] \in \Sigma_C \quad \text{if } (0, x_2, \zeta) \neq (0, 0, a\omega - a). \quad (7.5)$$

Similarly using the local coordinates $(\zeta, x_2, s) \mapsto [1 : s\zeta : x_2 : -a - \omega x_2 + s]$ in Γ_{11} , we have

$$f_{Z_1} : \Gamma_{11} \ni (\zeta, x_2, 0) \mapsto [\zeta : x_2\zeta : \zeta(-a - \omega x_2) : 1] \in \Sigma_C \quad \text{if } (\zeta, x_2, 0) \neq (\frac{1}{a\omega - a}, 0, 0). \quad (7.6)$$

Since both $(0, 0, a\omega - a)$ in (7.5) and $(1/(a\omega - a), 0, 0)$ in (7.6) correspond to the point $\gamma_{11} \cap \gamma_{10}$, combining with the previous conversation about $\Gamma_{11} - \pi_1^{-1}(\gamma_{11} \cap \gamma_{10})$, we conclude that f_{Z_1} is regular at all points of $\Gamma_{11} - (\gamma_{10} \cup \mathcal{F}_{0\beta\gamma}^1)$. \square

Lemma 7.3. *The three curves $\gamma_5, \gamma_{11}, \mathcal{F}_{0\beta\gamma}^1$ intersect transversally inside Y , and $\gamma_1, \gamma_7, \mathcal{F}_{0BC}^1$ intersect transversally inside Y . Thus, inside Z_1 , the strict transform of $\mathcal{F}_{0\beta\gamma}^1$ is disjoint from the strict transforms of γ_j , $1 \leq j \leq 10$.*

Proof. It suffices to prove the first statement. We may write $\gamma_{11} \subset \mathbf{P}^3$ as $s \mapsto [s : 0 : 1 : -as - \omega]$. This intersects Σ_{01} in the point $[0 : 0 : 1 : -\omega]$. We use the coordinate system $\pi : (u, \eta, x_3) \mapsto [u : u\eta : 1 : x_3] \in \mathbf{P}^3$. Thus $\mathcal{F}_{0\beta\gamma}^1 = \{u = 0, x_3 = -\omega\}$. In this coordinate system, γ_{11} becomes $s \mapsto (s, 0, -as - \omega)$, so γ_{11} crosses S_{01} when $s = 0$, at the point $(0, 0, -\omega)$. On the other hand, if we map γ_{11} backward under f_Y^{-6} , we find an expression for γ_5 . The base point is given by $[0 : 0 : 1 : s]$, and the fiber coordinate is given by $\eta = (1 + as)(1 + as + \omega(1 + a - s))/(as(-1 + s + as - \omega(1 + a - as)))$. Thus when the base point is $[0 : 0 : 1 : -\omega]$, we have $\eta = 0$. Thus all three curves meet at $(u, \eta, x_3) = (0, 0, -\omega)$. The curve γ_{11} is transverse to S_{01} , but γ_5 and $\mathcal{F}_{0\beta\gamma}^1$ are tangential to S_{01} , so γ_{11} is transverse to the other two, and γ_5 is transverse to $\{x_3 = -\omega\}$, while $\mathcal{F}_{0\beta\gamma}^1$ is tangential to this set. \square

For $2 \leq j \leq 11$, let $\pi_j : Z_j \rightarrow Z_{j-1}$ be the blowup of the strict transform of γ_{12-j} inside Z_{j-1} and $\pi : Z \rightarrow Y = \pi_{11} \circ \pi_{10} \circ \cdots \circ \pi_1$, that is, we blowup γ_{11} first, then γ_{10} , then γ_9 , etc. Let $f_{Z_j} : Z_j \dashrightarrow Z_j$, $f_Z : Z \dashrightarrow Z$ denote the induced map.

Lemma 7.4. *For $1 \leq j \leq 10$, $\mathcal{I}(f_{Z_j}) = \mathcal{F}_{0\beta\gamma}^1 \cup \Sigma_{02} \cup \gamma_{11-j}$.*

Proof. Suppose p is a point of $\gamma_j \cap \gamma_k$ $1 \leq j < k \leq 10$. Because of the order of blowup, γ_k is blown up before γ_j and γ_{k+1} is blown up before γ_{j+1} . Since f_Y is regular at p and the order of blowups at p is consistent with the order of blowups at $f_Y(p)$, the induced map f_{Z_i} is a local biholomorphism in a neighborhood of the exceptional divisor over p for $12 - j \leq i \leq 11$.

Notice that for all $1 \leq j \leq 11$ the strict transformation of γ_j does not intersect Σ_{02} in Y . Suppose γ_j intersects $\mathcal{F}_{0\beta\gamma}^1$ at a point q . Using the local coordinates in the neighborhood $(s, \zeta, x_3)_{S_{01}}$, we may assume that $q = (0, \zeta_*, -\omega)_{S_{01}}$ and $\gamma_j(s) = (Q_1(s), Q_2(s) + \zeta_*, Q_3(s) - \omega)_{S_{01}}$, where $\gamma_j = \{\gamma_j(s), s \in \mathbf{C}\}$, and $\gamma_j(0) = q$. Consider two local coordinate charts covering the exceptional divisor over the point q :

$$\begin{aligned} (s, t, \eta) &\mapsto (Q_1(s), Q_2(s) + \zeta_* + t, Q_3(s) - \omega + t\eta)_{S_{01}} \\ (s, \eta, t) &\mapsto (Q_1(s), Q_2(s) + \zeta_* + t\eta, Q_3(s) - \omega + t)_{S_{01}} \end{aligned}$$

With a computation similar to Lemma 7.2, We see that the induced map is regular everywhere on the exceptional divisor over q , $\mathcal{F}(q)$, except the point of intersection $\mathcal{F}(q) \cap \mathcal{F}_{0\beta\gamma}^1$. Now since the curve γ_{11-j} is the pre-image of γ_{12-j} , we have $\mathcal{I}(f_{Z_j}) = \mathcal{F}_{0\beta\gamma}^1 \cup \Sigma_{02} \cup \gamma_{11-j}$. \square

From the previous Lemma we have $\mathcal{I}(f_{Z_{10}}) = \mathcal{F}_{0\beta\gamma}^1 \cup \Sigma_{02} \cup \gamma_1$. Since Σ_γ is the pre-image of γ_1 , we have $\mathcal{I}(f_Z) \subset \mathcal{F}_{0\beta\gamma}^1 \cup \Sigma_{02} \cup \Sigma_\gamma$. From (5.2) we see that for all most every line $\ell \subset \Sigma_\gamma$, through e_1 in Σ_γ , f maps ℓ regularly to a point $q \in \gamma_1$. In our construction of Z , we blew up $\gamma_{11}, \dots, \gamma_2$ before γ_1 . Thus the map f_Z will map ℓ regularly to the fiber of Γ_1 over q unless q is an intersection point of $\gamma_1 \cap \gamma_j$ for some $2 \leq j \leq 11$.

Lemma 7.5. *Suppose $q \in \gamma_1 \cap \gamma_j$ for some $j = 2, \dots, 11$ and $\ell_j \subset \Sigma_\gamma$ be the line which mapped to q by f_Y . The line $\ell_j \subset \mathcal{I}(f_Z)$ and every point in ℓ_j blows up to the first blowup fiber \mathcal{F}_q^1 .*

Proof. Let us parametrize $\gamma_1 = \{\gamma_1(t) = [-\frac{1}{a}(1+\omega t) : t : 1 : 0], t \in \mathbf{C}\}$. Let us set $q = \gamma_1(t_*)$ for some $t_* \in \mathbf{C}$ and $\gamma_j = \{\gamma_j(s) = [Q_0(s) - \frac{1}{a}(1+\omega t_*) : Q_1(s) + t_* : Q_2(s) + 1 : Q_3(s)]\}$. The line ℓ_j is given by the strict transform in Y of the line connecting e_1 and $\tilde{q} = [-\frac{1}{a}(1+\omega t_*) : 0 : t_* : 1]$ in \mathbf{P}^3 . To see the image of the line ℓ_j , we consider the set $U = \{[-\frac{1}{a}(1+\omega t_*) + s\zeta : u : t_* + s : 1]\}$ which has the property that $U \cap \{s = 0\} = \ell_j - \{e_1\}$. Since the point q is blown up twice, let us consider a local coordinate charts for $\pi_{12-j}^{-1}(\gamma_j)$:

$$(v, \xi, s)_{\gamma_j} \mapsto \left[\frac{Q_0(s) - \frac{1}{a}(1+\omega t_*)}{Q_2(s) + 1} + v : \frac{Q_1(s) + t_*}{Q_2(s) + 1} + v\xi : 1 : \frac{Q_3(s)}{Q_2(s) + 1} \right].$$

Using the induced map f_Z , we see that

$$f_Z : \ell_j \ni [-\frac{1}{a}(1+\omega t_*) : u : t_* : 1] \rightsquigarrow \{(0, \xi, 0), \xi \in \mathbf{C}\} \subset \Gamma_j,$$

that is, each point in ℓ_j blows up to a whole first blowup fiber over q . \square

Before Lemma 7.2, we enumerated the possibilities for lines ℓ and points q as in the hypotheses of Lemma 7.5. Thus we may combine Lemmas 7.2–5 to have the following Theorem:

Theorem 7.6. *The indeterminacy locus $\mathcal{I}(f_Z) = \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}^1 \cup \ell_2 \cup \ell_3 \cup \ell_7 \cup \ell'_9 \cup \ell''_9$. If ζ is a point of one of the lines ℓ , then f_Z blows up ζ to the first fiber $\mathcal{F}_{f(\ell)}^1$.*

Now we give the existence of Green currents for the invariant class $\alpha = \alpha_Z^+ \in H^{1,1}(Z)$.

Theorem 7.7. *There is a positive closed current T_Z^+ in the class of α_Z^+ with the property: if Ξ^+ is a smooth form which represents α_Z^+ , then $\lim_{n \rightarrow \infty} \delta_1(f)^{-n} f_Z^{n*} \Xi^+ = T_Z^+$ in the weak sense of currents on Z .*

Proof. The map f_Z^* is given in Appendix A, where we are in Case (II). Working directly with the matrix (A.1), we see that the invariant class is given by:

$$\alpha = H_Z - c_1 \tilde{E}_1 - c_3 \tilde{E}_3 - c_{01} \tilde{S}_{01} - c_{03} \tilde{S}_{03} - \sum_{j=1}^{11} c'_j \mathcal{F}_j$$

where $c_1, c_3 > 0$, $c_1 + c_3 = 1$, $c'_{11} > c'_{10} > \dots > c'_1 > 0$, and $c_{01} = c_{03} > c'_8$. As in Theorem 4.9, we will show that $\alpha_Z^+ \cdot \sigma$ for each curve σ inside the forward image of $\mathcal{I}(f_Z)$. The result will follow from Theorem 1.3 of [Ba].

Let us start with $\mathcal{F}_{0\beta\gamma} \subset \mathcal{I}(f_Z)$. Points of this curve are blown up to \mathcal{F}_{0BC} . The curve $\sigma = \mathcal{F}_{0BC}$ is the exceptional fiber inside S_{03} over the point $\Sigma_{BC} \cap \Sigma_{03} \in \mathbf{P}^3$. Thus $\sigma \cdot S_{03} = -1$. In the construction of Z , γ_7 will be blown up to create the exceptional divisor Γ_7 . At this stage, by Lemma 7.3, σ and γ_1 become separated. Thus $\sigma \cdot \Gamma_1 = 0$, and $\sigma \cdot \Gamma_7 = 1$, so, so $\alpha \cdot \sigma = c_{03} - c_7 > 0$.

Points of the indeterminate curve Σ_{02} blow up to $\sigma = \mathcal{F}_{e_2}^1$. In this case, we have that $\sigma \cdot S_{01}$ and $\sigma \cdot S_{03}$, are ± 1 , with opposite signs, so $\sigma \cdot \alpha_Z^+ = \pm c_{01} \mp c_{03} = 0$ as was seen in the proof of Theorem 4.9.

The other possibility is $\ell \subset \mathcal{I}(f_Z)$, for one of the indeterminate lines in Σ_γ . This blows up to one of the first fibers $\sigma = \mathcal{F}_\zeta^1$. In this case, σ crosses Γ_1 transversally, so $\sigma \cdot \Gamma_1 = 1$. On the other hand, $\sigma \subset \Gamma_j$ for some $j > 1$, so we have $\sigma \cdot \Gamma_j = -1$. Thus $\sigma \cdot \alpha_Z^+ = c'_j - c'_1 > 0$. \square

Remark. Considering the symmetry between f and f^{-1} , we find that $\mathcal{I}(f_Z^{-1}) = \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}^1 \cup \bigcup_\zeta \mathcal{F}_\zeta^1$, where the ζ 's are the intersection points of γ_1 with the curves $\gamma_2, \gamma_3, \gamma_7$, and γ_9 .

If we instead blow up the γ_j 's in the order $\gamma_1, \gamma_2, \dots$, and call the resulting space \hat{Z} . Then we have $\mathcal{I}(f_{\hat{Z}}) = \Sigma_{02} \cup \mathcal{F}_{0\beta\gamma}^1 \cup \bigcup_\zeta \mathcal{F}_\zeta^1$, where the $\zeta \in \gamma_{11}$ are the points of intersection with $\gamma_3, \gamma_5, \gamma_9$, and γ_{10} . Each of these points ζ is blown up by $f_{\hat{Z}}$ to a line of the pencil in Σ_C passing through e_3 .

Thus we can apply a similar argument to α_Z^- to obtain the Green current for f_Z^{-1} .

Corollary 7.8. *There is a positive closed current T_Z^- in the class of α_Z^- with the property: if Ξ^- is a smooth form which represents α_Z^- , then $\lim_{n \rightarrow \infty} \delta_1(f)^{-n} f_Z^{n*} \Xi^- = T_Z^-$ in the weak sense of currents on Z .*

Next we show what happens to the invariant fibration when we lift it to Z . Let us set $P_0 = x_0 x_1 x_2 x_3$, and let P_1 be a homogeneous quartic polynomial defined in Appendix B. For $c \in \mathbf{C}$, let us set $S_c = \{cP_0 + P_1 = 0\}$, so the rotor \mathcal{R} corresponds to $c = \infty$. Since we have $f(S_c) = S_{\omega c}$, the surface S_0 is invariant.

Proposition 7.9. *The variety $S_0 := \{P_1 = 0\} \subset \mathbf{P}^3$ has singular points at e_1, e_3 and the fixed points p_\pm . If p_\pm are blown up (in addition to the e_1 and e_3 which were blown up to construct Y), then the strict transform of S_0 is a nonsingular $K3$ surface.*

Proof. Using the computer, we find that the critical points of P_1 occur exactly at e_1, e_3 and $p_\pm = (x_\pm, x_\pm, x_\pm) \in \mathbf{C}^3$ where x_\pm are the roots of $x^2 = a + (1 + \omega)x$. (*Mathematica*, for

instance, can do this.) Further, p_{\pm} are singular points of type A_1 . The singular points e_1 and e_3 are type A_1 unless $a = (1 + 2\omega)/(1 - \omega)$, in which case they are type A_2 . In either case, it follows (see, for instance, [EJ, Lemma 3.1 and Remark 3.2]) that S_0 is $K3$. \square

Corollary 7.10. *For all but finitely many values of $c \in \mathbf{C}$, the strict transform of S_c in Z is a nonsingular $K3$ surface.*

Let $\mathcal{P} \subset Y$ denote the (finite) set of all intersection points of distinct curves $\gamma_j \cap \gamma_k$. Since the γ_j lie in the rotor, we have $\mathcal{P} \subset \mathcal{R}$. The rotor is the union of 8 smooth hypersurfaces which intersect transversally, so the singular locus of \mathcal{R} is the set where two (or more) of these surfaces intersect. We will write \mathcal{P}_s (resp. \mathcal{P}_r) for the points of \mathcal{P} which are contained in the singular (resp. regular) locus of \mathcal{R} .

While Z itself depends on the order in which the curves γ_j are blown up, the following Propositions are valid for any ordering of the blowups.

Proposition 7.11. *For $p \in \mathcal{P}_r$, there is a unique $c_p \in \mathbf{C}$ such that $S_{c_p} \subset Y$ is singular at p . This is a conical singularity, and the strict transform $S_{c_p} \subset Z$ contains the first fiber $\mathcal{F}_{c_p}^1$.*

Proof. Without loss of generality, we may choose coordinates (x, y, z) so that $p = 0$, $L = z$ near p , and $\mathcal{R} = \{z = 0\}$. Let us suppose that $p \in f_Y^j \Sigma_{BC} \cap f_Y^k \Sigma_{BC}$. Since the curves $f_Y^j \Sigma_{BC}$ are contained in \mathcal{R} and intersect transversally, we may suppose that near p the curves $f_Y^j \Sigma_{BC}$ and $f_Y^k \Sigma_{BC}$ coincide with the x - and y -axes. Thus the tangent to $\{M = 0\}$ at p is given by $z = 0$, so we may suppose that $M = \lambda z + xy + \dots$. The surfaces are then $S_c = \{M + cL = 0\} = \{\lambda z + xy + cz + \dots = 0\}$. The surface S_c is singular if $c = -\lambda$. We blow up the x -axis by the coordinate change $(x, s, \eta) \mapsto (x, s, s\eta)$. The first fiber is $\mathcal{F}_p^1 = \{x = s = 0\}$. The strict transforms of the surfaces are $S_c = \{(\lambda + c)\eta + x = 0\}$. The strict transform of the y -axis is now the s -axis, which is contained in each S_c . Otherwise, the S_c 's are disjoint. The strict transform of $S_{-\lambda}$ contains \mathcal{F}_p^1 . After we blow up the s -axis, the surfaces are all disjoint and smooth. \square

Proposition 7.12. *For $p \in \mathcal{P}_s$, S_c is smooth at p for all $c \in \mathbf{C}$. The first fiber is contained in the rotor: $\mathcal{F}_p^1 \subset \mathcal{R} \subset Z$.*

Proof. We may assume that p is a normal crossing of two of the hypersurfaces of \mathcal{R} . Thus we may choose coordinates (x, y, z) such that $p = 0$, and $L = xy$ near p . We may assume that $f_Y^j \Sigma_{BC}$ is the x -axis, and $f_Y^k \Sigma_{BC}$ is the y -axis. Since M contains both axes, we may assume that $M = z + \varphi$, where φ is divisible by xy . Thus $S_c = \{M + cL = z + \varphi + cxy = 0\}$ is smooth for all $c \in \mathbf{C}$. When we blow up the x -axis, we use coordinates $(x, s, \eta) \mapsto (x, s, s\eta)$. The strict transforms are then $S_c = \{s\eta + \tilde{\varphi} + csx = 0\}$, where $\tilde{\varphi}$ is divisible by xs . Dividing this equation by s , we have $S_c = \{\eta + \psi(x, s, \eta) + cx = 0\}$, where $\psi(0, s, 0) = 0$, since S_c contains the s -axis (the strict transform of the y -axis). We have $\mathcal{F}_p^1 = \{x = s = 0\}$. Now we blow up the s -axis via the coordinates $(\xi, s, t) \mapsto (\xi t, s, t) = (x, s, \eta)$. This gives the new strict transforms $S_c = \{1 + \hat{\psi}(\xi, s, t) + c\xi = 0\}$, where $\hat{\psi}(\xi, s, t) = t^{-1}\psi(\xi t, s, t)$ is regular. The strict transform of \mathcal{F}_p^1 is now $\{\xi = s = 0\}$, which is disjoint from the S_c s. \square

If $p' \in \gamma_1 \cap \gamma_9$, then there is a unique $c' \in \mathbf{C}$ such that $S_{c'}$ is singular at p' . Let ℓ'_9 denote the line for which $f(\ell'_9) = p'$. By Theorem 7.6 and Proposition 7.11, it follows that f_Z maps ℓ'_9 to the strict transform of $S_{c'}$ inside Z . Thus the total transform of ℓ'_9 under f_Z^n is contained in $S_{\omega^n c'}$. Let p'' denote the other point of $\gamma_1 \cap \gamma_9$, and let $c'' \in \mathbf{C}$ denote the corresponding parameter. Let $\hat{S} = S_{c'} \cup S_{\omega c'} \cup S_{\omega^2 c'} \cup S_{c''} \cup S_{\omega c''} \cup S_{\omega^2 c''}$, we see that \hat{S} is a

f_Z -invariant set which contains $\ell'_9 \cup \ell''_9$. Let \mathcal{R} denote the strict transform of the rotor in Z . The sets \mathcal{R} , \hat{S} , and $\Sigma_{02} \cup \mathcal{F}_{e_2}^1$ are totally invariant, and we break the indeterminacy locus into three sets:

$$\mathcal{I}(f_Z) = (\Sigma_{02} \cup \mathcal{F}_{e_2}^1) \cup (\mathcal{I}(f_Z) \cap \mathcal{R}) \cup (\mathcal{I}(f_Z) \cap \hat{S})$$

with $\mathcal{I}(f_Z) \cap \mathcal{R} = \ell_2 \cup \ell_3 \cup \ell_7 \cup \mathcal{F}_{0\beta\gamma}^1$, and $\mathcal{I}(f_Z) \cap \hat{S} = \ell'_9 \cup \ell''_9$. We set

$$\Omega = Z - (\hat{S} \cup \mathcal{R} \cup \Sigma_{02} \cup \mathcal{F}_{e_2}^1)$$

By Propositions 7.11 and 7.12, \mathcal{R} is disjoint from the strict transform of each S_c . Thus f_Z is regular on Ω , and Ω is invariant under f_Z .

Proposition 7.13. *For every S_c in Ω the dynamical degree of the restriction is $\delta(f_c^3) = \delta_1(f)^3$.*

Proof. Let us denote Γ a hypersurface in Z whose cohomology class in $H^{1,1}(Z)$ is H_Z . It follows that the degree of $f_Z^{-3n}\Gamma$ grows like $\delta_1(f)^{3n}$. On the other hand $S_c \subset \Omega$ does not contain an irreducible component of the indeterminacy locus for f_Z . It follows that we have $S_c \cap (f_Z^3)^{-n}\Gamma = (f_Z^3)^{-n}(S_c \cap \Gamma)$. Because S_c is non-singular and f_Z^3 is pseudo-automorphism, the degree of $(f_Z^3)^{-n}(S_c \cap \Gamma) = (f_c^3)^{-n}(S_c \cap \Gamma)$ is $4\delta_1(f)^{3n}$. Thus the dynamical degree of f_c^3 is $\delta_1(f)^3$. \square

Using the fact that f_Z is regular on the large invariant set Ω , we avoid the difficulties that can occur in defining the entropy of a map (see [G1]).

Theorem 7.14. *The entropy of f is $\log \delta_1(f)$.*

Proof. Since f is equivalent to a pseudo automorphism, and f_Z^* is conjugate to $(f_Z^*)^{-1}$, both the first and the second dynamical degrees are equal. Combining the result in [DS] and the fact that $h_{top}(f) \geq h_{top}(f_0)$, we have the inequality

$$\log \delta_1(f) \geq h_{top}(f) \geq h_{top}(f_0) = \log \delta_1(f)$$

which gives the result. \square

Since Ξ^\pm and f_Z are regular on Ω , the potential of g^\pm is continuous on Ω . Thus may define the wedge product $T_2 := T^+ \wedge T^-$ as a positive, closed (2,2)-current on Ω , and we have:

Proposition 7.15. $\lim_{n \rightarrow \infty} \delta_1(f)^{-2n} f_Z^{*n} \Xi^+ \wedge f_Z^{*-n} \Xi^- = T_2$ exists as a (2,2)-current on Ω .

We have seen that the restrictions $f^3|_{S_c}$ are automorphisms, and there are invariant currents μ_c^\pm on S_c , as well as invariant measures $\mu_c := \mu_c^+ \wedge \mu_c^-$ (see [C]). The following property leads us to consider T^\pm and T_2 as the “bifurcation currents” for the family $\{f^3|_{S_c}\}$ (see [DuF]).

Theorem 7.16. *For $S_c \subset \Omega$ the slices by S_c are well-defined and give the corresponding dynamical objects: $T^\pm|_{S_c} = \mu_c^\pm$, and $T_2|_{S_c} = \mu_c$.*

Proof. If we set $h = f^3$, then the class $[S_c]$ is invariant under h^* . Thus $\alpha^+ \cdot [S_c] \in H^{1,1}(S_c)$ is a class that is expanded by a factor of $\delta_1(f)$. It follows that the restriction $\Xi^+|_{S_c}$ gives the expanded class, and this converges to μ_c^+ . Similarly, the normalized pullbacks/push-forwards of $\Xi^+ \wedge \Xi^-$ on S_c will converge to μ_c . \square

Theorem 7.17. *For generic c', c'' , the maps $f^3|_{S_{c'}}$ and $f^3|_{S_{c''}}$ are not smoothly conjugate, and the surfaces $S_{c'}$ and $S_{c''}$ are not isomorphic.*

Proof. There is an invariant 6-cycle of curves, Γ_j , $j = 0, \dots, 5$ for f . For generic c , $\Gamma_j \cap S_c$ is a saddle 2-cycle for $f^3|_{S_c}$. The multipliers of this saddle cycle are not constant in c , so the maps $f^3|_{S_c}$ are not smoothly conjugate. Since the automorphism group of S_c is disconnected, we see that the family $\{S_c\}$ cannot consist of surfaces which are all isomorphic to each other. \square

Remark. If $a_2 \neq 1$ is a primitive 5th root of unity and $a_0 = b_0 = 0$, then we may repeat most of the arguments in this section for this map. In particular, we have:

Theorem 7.18. *If a_2 is a primitive 5th root of unity, and $a_0 = b_0 = 0$, then f is equivalent to a pseudo-automorphism, and the dynamical degrees $\delta_1(f) = \delta_2(f) \approx 1.3211018 > 1$ are the largest root of $t^{19}(t^3 - t - 1) + t^3 + t^2 - 1$. The entropy of f is $\log \delta_1(f) > 0$. Furthermore there are two quartic polynomials which are invariant in the sense of (B.1). This gives a family of K3 surfaces which are invariant under f^5 .*

§8. Pseudo-automorphisms which are completely integrable. Let us consider two cases for maps of the form (2.2):

$$\alpha = (a, 0, 1, 1), \quad a \neq 1, \quad \text{and} \quad \beta = (0, 1, 0, 0) \quad (8.1a)$$

$$\alpha = (0, 0, \omega, 1), \quad \omega^3 = 1, \quad \omega \neq 1, \quad \text{and} \quad \beta = (0, 1, 0, 0) \quad (8.1b)$$

The map (8.1a) has been extensively studied under the name Lyness process. The maps (8.1a) and (8.1b) exhibit similarities to the maps in the previous section: they are critical maps, and the iterates of the critical image Σ_{BC} go “once around” the rotor and land on $\Sigma_{\beta\gamma}$. The difference with §7 is that $f_Y^4 \Sigma_{BC} = \mathcal{F}_{0\beta\gamma}$ is an indeterminate curve, and by Lemma 4.4 this fiber is mapped to \mathcal{F}_{0BC} , that is, $f_Y^4 \Sigma_{BC} = \mathcal{F}_{0\beta\gamma} \subset S_{01}$ and $f_Y^5 \Sigma_{BC} = \mathcal{F}_{0BC} \subset S_{03}$. Thus Σ_{BC} arrives at $\Sigma_{\beta\gamma}$ one step faster than was the case in §7.

Let $\pi : Z \rightarrow Y$ denote the space obtained by blowing up the orbit $f^j \Sigma_{BC}$, $f^{-j} \Sigma_{\beta\gamma}$, $0 \leq j \leq 4$ (one curve less than the construction in §7).

Theorem 8.1. *The induced map f_Z is a pseudo-automorphism, and the iterates of f have quadratic degree growth.*

Proof. Since $f_Y^9 \Sigma_{BC} = \Sigma_{\beta\gamma}$ and $f_Y^4 \Sigma_{BC} = \mathcal{F}_{0\beta\gamma}$, $f_Y^5 \Sigma_{BC} = \mathcal{F}_{0BC}$, we see that f_Y satisfies the condition in Theorem 5.1. This Theorem then follows from Theorems 5.1 and 5.3. \square

Proposition 8.2. *In cases (8.1a) and (8.1b), the induced rotor map $f^8|_{\Sigma_3}$ has linear degree growth. This map is not birationally conjugate to a surface automorphism.*

Proof. In the case (8.1b), the restriction of f_Y^8 to Σ_3 is given by setting $a_0 = 0$ in (7.3), so we find the degree 2 birational map :

$$f_Y^8|_{\Sigma_3} : [x_0 : x_1 : x_2 : 0] \mapsto [x_0 \omega^2 x_2 : x_1(x_1 + \omega^2 x_0 + \omega^2 x_2) : \omega^2 x_2(x_1 + \omega^2 x_2) : 0]$$

This map has three distinct exceptional lines. Two of exceptional lines are mapped to fixed points $[1 : 0 : -1 : 0]$ and $[0 : 1 : -a : 0]$. The remaining exceptional line is mapped to a point of indeterminacy $e = [1 : -1 : 0 : 0]$. We let W be the blowup space obtained by blowing up Σ_3

at e . The induced map has only two exceptional lines which are mapped to fixed points and therefore the induced map is algebraically stable. The action on Pic is given by the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ which has an eigenvalue 1 with 2×2 jordan block. It follows that the degree of restriction map grows linearly.

The analysis in the case (8.1a) is essentially the same. The induced rotor map is now:

$$f_Y^8|_{\Sigma_3} : [x_0 : x_1 : x_2 : 0] \mapsto [x_0(ax_0 + ax_1 + x_2) : x_1(x_0 + x_1 + x_2) : x_2(ax_0 + x_1 + x_2) : 0]$$

This map has three exceptional lines. Two of them are mapped to fixed points $[1 : 0 : -1]$ and $[0 : 1 : -a]$. The third exceptional line is mapped to $[1 : -1 : 0]$, which is indeterminate. After we blow up the point $[1 : -1 : 0]$, the induced map is algebraically stable and the action on Pic has an eigenvalue 1 with 2×2 jordan block.

Finally, since the restriction of f_W to the rotor has linear degree growth. It follows from [DiF] that this restriction is not an automorphism. \square

We consider first the Lyness map, i.e., case (8.1a). This is known to be integrable, and the invariant polynomials are given in [CGM1] and [KoL]. These invariant polynomials, which satisfy (B.1) with $t = 1$, are:

$$\begin{aligned} Q_0 &= x_0 x_1 x_2 x_3 \\ Q_1 &= (ax_0 + x_1 + x_2 + x_3)(x_0 + x_1)(x_0 + x_2)(x_0 + x_3) \\ Q_2 &= (x_0(ax_0 + x_1 + x_2 + x_3) + x_1 x_3)(x_0 + x_1 + x_2)(x_0 + x_2 + x_3). \end{aligned} \tag{8.2a}$$

The set $\{Q_0 = 0\}$ gives an invariant 8-cycle of rational surfaces, which is the rotor $\mathcal{R} \subset Y$. (Although $Q_0 = 0$ consists of 4 irreducible components in \mathbf{P}^3 , it yields an 8-cycle inside Y because these components map through the indeterminacy locus, which is blown up to yield an additional 4 divisors.) The set $\{Q_1 = 0\}$ gives an invariant 4-cycle, and $\{Q_2 = 0\}$ gives an invariant 3-cycle; the components of the 8-, 4-, and 3-cycles are rational surfaces. As we observed in §4, f_Y induces dominant maps on each of these cycles. And as in Proposition 8.2, we may show that the restriction of f^4 to the 4-cycle, and the restriction of f^3 to the 3-cycle both have linear degree growth.

Let us define the surfaces $S_c = \{Q_c = 0\}$ with $Q_c := c_0 Q_0 + c_1 Q_1 + c_2 Q_2$. If we also write S_c for its strict transform inside Z , we have $f S_c = S_c$.

Theorem 8.3. *For generic c , the surface S_c is an irreducible $K3$ surface.*

Proof. For generic c , we find that S_c has 16 singular points: two of them are e_1, e_3 , which are type A_2 , and there are 14 more which are of type A_1 . In the construction of Z , we blew up e_1 and e_3 . Then we blew up $f^j \Sigma_{BC}$, $0 \leq j \leq 10$, and the other 14 singular points are contained in these curves. It follows that the strict transform of S_c inside Z is smooth and thus $K3$. \square

Theorem 8.4. *For generic c and c' , the intersection $S_c \cap S_{c'}$ is an elliptic curve. The restriction of f^3 to S_c has quadratic degree growth.*

Proof. Since S_c is a $K3$ surface, it has trivial canonical bundle. Thus the birational map f^3 of S_c must be an automorphism. For generic c and $c' \neq c$, the intersections $S_c \cap S_{c'}$ give an invariant fibration of S_c . Since $f^3|_{S_c}$ is an automorphism, then by [DiF] the intersection $S_c \cap S_{c'}$ is an elliptic curve and the restriction of f to the family of $K3$ surfaces has quadratic degree growth. \square

The map (8.1b) is similar. In this case the solutions to (B.1) take the form:

$$\begin{aligned}
R_0 &= x_0 x_1 x_2 x_3 \\
R_1 &= (x_0 + \omega x_1)(x_0 + \omega x_2)(x_0 + \omega x_3)(x_1 + \omega^2 x_2 + \omega x_3) \\
R_2 &= \omega x_1 x_3 (x_0 + \omega x_1)(x_0 + \omega x_3) \\
&\quad + \omega^2 x_0 x_2 (x_0(x_1 + \omega x_3) + x_2(\omega x_1 + x_3) + \omega^2 x_0 x_2)
\end{aligned} \tag{8.3}$$

where $t_{R_0} = 1$, $t_{R_1} = \omega^2$, and $t_{R_2} = \omega^2$. As before, we see that f_Z will have an invariant 8-cycle given by the rotor $\mathcal{R} \subset Z$. And $\{R_1 = 0\}$ will give a 4-cycle of rational surfaces. For generic c , the singularities of the surface $S_c = \{\sum c_j R_j = 0\}$ are e_1, e_3 (type A_2) and e_2 (type A_1). As in Theorems 8.3 and 8.4, we have:

Theorem 8.5. *In case (8.1b): for generic c , S_c is a K3 surface, f^3 is an automorphism of S_c with quadratic growth, and the intersections $S_c \cap S_{c'}$ are elliptic curves.*

§A. Appendix: Computing the Characteristic Polynomial for f_Z^* . Let us consider a critical map f satisfying condition (5.1), and let m_u, m_d, m_s, d_j, u_j and N be the numbers defined in §5. We define the $(N+5) \times (N+5)$ matrix

$$\begin{pmatrix}
2 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & -1 & 1 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & -1 \\
* & & & & & & & & * \\
\vdots & & & & & & & & \vdots \\
* & & & & & & & & *
\end{pmatrix} \tag{A.1}$$

where the $*$'s indicate that the 7th through the $N+5$ th rows remain to be specified. We will define the j th row r_j in terms of the elements e_k , which are vectors of length $N+5$ in which the k th entry is 1, and all other entries are 0:

- (a) if $j = N - d_i$ for some $i = 1, \dots, m_d$, then $r_{j+6} = e_{j+5} - e_{N+5}$
- (b) if $j = N - u_i$ for some $i = 1, \dots, m_u$, then $r_{j+6} = -e_1 - e_5 + e_{j+5} - e_{N+5}$
- (c) if $j = N - m_s$,

$$\begin{aligned}
r_{j+2} &= -e_1 - e_3 + e_{j+1} - e_{N+5}, & r_{j+3} &= -e_3 + e_{j+2} \\
r_{j+4} &= -e_1 - e_3 - e_5 + e_{j+3} - e_{N+5}, & r_{j+5} &= -e_1 - e_3 - e_5 + e_{j+4} \\
r_{j+6} &= -e_5 + e_{j+5}
\end{aligned}$$

- (d) Otherwise, $r_{j+6} = e_{j+5}$.

Let $\pi : Z \rightarrow \mathbf{P}^3$ be the space constructed in Theorems 5.1 and 5.3.

Proposition A.1. *The matrix (A.1) represents f_Z^* .*

Proof. There are three cases to consider. Although we define a different basis in each case, the matrix (A.1) representing f_Z^* is the same.

Case (I) : There is no $1 \leq j \leq N$ such that $f_Y^j \Sigma_\gamma \subset \mathcal{F}_{0\beta\gamma} \cup \Sigma_{\beta\gamma}$.

In this case for all $j \neq d_i, u_k, m_s, m_{s+1}, \dots, m_{s+4}$, $1 \leq i \leq m_d, 1 \leq k \leq m_u$ we have $\pi \mathcal{F}_j \not\subset \Sigma_0 \cup \Sigma_\beta \cup \Sigma_\gamma$ and therefore

$$\begin{aligned} f_Z^* H_Z &= 2\mathcal{H}_Z - E_1 - S_{01} - E_3 - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} - \mathcal{F}_{m_s+4} \\ \{\Sigma_0\} &= \mathcal{H}_Z - E_1 - S_{03} - S_{01} - E_3 - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} - \mathcal{F}_{m_s+3} - \mathcal{F}_{m_s+4} \\ \{\Sigma_\beta\} &= \mathcal{H}_Z - S_{01} - E_3 - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} \\ \{\Sigma_\gamma\} &= \mathcal{H}_Z - E_1 - \mathcal{F}_N - \mathcal{F}_{m_s+1} - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \sum_{i=1}^{m_d} \mathcal{F}_{d_i}. \end{aligned}$$

Since we have $f_Z^* : E_1 \mapsto S_{03} \mapsto \{\Sigma_0\}$, $S_{01} \mapsto E_3 \mapsto \{\Sigma_\beta\}$, $\mathcal{F}_j \mapsto \mathcal{F}_{j-1}$ for all $j = 2, \dots, N$, and $\mathcal{F}_1 \mapsto \{\Sigma_\gamma\}$ using the ordered basis $\{H_Z, E_1, S_{03}, S_{01}, E_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \dots, \mathcal{F}_2, \mathcal{F}_1\}$ for $\text{Pic}(Z)$ we see that (A.1) is the matrix representation for f_Z^* .

Case (II) : There are κ positive integers $1 < s_1 < \dots < s_\kappa < N$ such that for $j = 1, \dots, \kappa$ $f_Y^{s_j} \Sigma_\gamma \subset \Sigma_\beta \setminus \ell_\beta \cup \Sigma_{\beta\gamma} \cup \ell'_\beta$ where $f_Y^3 \ell'_\beta = E_3 \cap \Sigma_0$.

For this case let us use the ordered basis $\tilde{\mathcal{B}} = \{H_Z, \tilde{E}_1, \tilde{S}_{03}, \tilde{S}_{01}, \tilde{E}_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \dots, \mathcal{F}_2, \mathcal{F}_1\}$ for $\text{Pic}(Z)$ where $\tilde{E}_3 = E_3 + \sum_{i=1}^\kappa \mathcal{F}_{s_i+1}$, $\tilde{S}_{01} = S_{01} + \sum_{i=1}^\kappa \mathcal{F}_{s_i+2}$, $\tilde{S}_{03} = S_{03} + \sum_{i=1}^\kappa \mathcal{F}_{s_i+4}$, and $\tilde{E}_1 = E_1 + \sum_{i=1}^\kappa \mathcal{F}_{s_i+5}$. Using this new ordered basis we can see that

$$\begin{aligned} f_Z^* : \tilde{E}_1 &\mapsto \tilde{S}_{03} \mapsto \{\Sigma_0\} + \sum_{i=1}^\kappa \mathcal{F}_{s_i+3} \\ &= \mathcal{H}_Z - \tilde{E}_1 - \tilde{S}_{03} - \tilde{S}_{01} - \tilde{E}_3 - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} - \mathcal{F}_{m_s+3} - \mathcal{F}_{m_s+4} \end{aligned}$$

In a similar way we may compute f_Z^* of $H_Z, \tilde{S}_{01}, \tilde{E}_3$ and \mathcal{F}_N and see that the matrix representation with $\tilde{\mathcal{B}}$ is given by (A.1).

Case (III) : There are τ positive integers $1 < q_1 < \dots < q_\tau < N$ such that $f_Y^{q_j} \Sigma_\gamma \subset \ell'_\beta$ for $j = 1, \dots, \tau$.

Let us consider the ordered basis $\hat{\mathcal{B}} = \{H_Z, \hat{E}_1, \hat{S}_{03}, \hat{S}_{01}, \hat{E}_3, \mathcal{F}_N, \mathcal{F}_{N-1}, \dots, \mathcal{F}_2, \mathcal{F}_1\}$ for $\text{Pic}(Z)$ where $\hat{E}_3 = \tilde{E}_3 + \sum_{i=1}^\tau (\mathcal{F}_{q_i+1} + \mathcal{F}_{q_i+3})$, $\hat{S}_{01} = \tilde{S}_{01} + \sum_{i=1}^\tau (\mathcal{F}_{q_i+2} + \mathcal{F}_{q_i+4})$, $\hat{S}_{03} = \tilde{S}_{03} + \sum_{i=1}^\tau (\mathcal{F}_{q_i+4} + \mathcal{F}_{q_i+6})$, and $\hat{E}_1 = \tilde{E}_1 + \sum_{i=1}^\tau (\mathcal{F}_{q_i+5} + \mathcal{F}_{q_i+7})$. Since $f_Y^2 \ell'_\beta = \Sigma_\beta \cap S_{01}$, we have

$$\begin{aligned} \{\Sigma_\beta\} &= \mathcal{H}_Z - \tilde{S}_{01} - \tilde{E}_3 - \sum_{i=1}^\tau (\mathcal{F}_{q_i} + \mathcal{F}_{q_i+1} + 2\mathcal{F}_{q_i+2} + \mathcal{F}_{q_i+3} + \mathcal{F}_{q_i+4}) \\ &\quad - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} \\ &= \mathcal{H}_Z - \hat{S}_{01} - \hat{E}_3 - \sum_{i=1}^\tau (\mathcal{F}_{q_i} + \mathcal{F}_{q_i+2}) - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2}. \end{aligned}$$

It follows that we have

$$\begin{aligned} f_Z^* : \hat{S}_{01} &\mapsto \hat{E}_3 \mapsto \{\Sigma_\beta\} + \sum_{i=1}^{\tau} (\mathcal{F}_{q_i} + \mathcal{F}_{q_i+2}) \\ &= \mathcal{H}_Z - \hat{S}_{01} - \hat{E}_3 - \mathcal{F}_N - \sum_{i=1}^{m_u} \mathcal{F}_{u_i} - \mathcal{F}_{m_s} - \mathcal{F}_{m_s+1} - \mathcal{F}_{m_s+2} \end{aligned}$$

For the other basis elements, computations are essentially identical and thus we see that (A.1) represents f_Z^* with respect to the ordered basis $\hat{\mathcal{B}}$. \square

According to the previous Proposition, we see that the characteristic polynomial of f_Z^* only depends on m_u, m_d, m_s, d_j, u_j and N .

Lemma A.2. *The characteristic polynomial of f_Z^* is given by*

$$\pm t^{N-1}(t^2 + 1) [(Q_1 - Q_4)t^3 + (2Q_1 - Q_2 - Q_3 - Q_4)t^2 + (Q_1 - Q_3)t + Q_4].$$

Proof. We subtract tI from the matrix (A.1) and perform a sequence of row operations on it. Step (i): we add or subtract the 6th row to the rows whose last entry is 1 or -1 and then (ii) for $j = 1, \dots, N-1$, we subtract $1/t^j$ times the $N+4-j$ th row from 6th row. This gives

$$\det(f_Z^* - tI) = \det \begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$$

where

$$A = \begin{pmatrix} 1-t & 0 & 1 & 0 & 0 & -t \\ 1-t & -t & 0 & 0 & 1 & 0 \\ 0 & 1 & -1-t & 0 & 0 & 0 \\ -1 & 0 & -1 & -t & -1 & 0 \\ -1 & 0 & -1 & 1 & -1-t & 0 \\ Q_1 & 0 & Q_2 & 0 & Q_3 & Q_4 \end{pmatrix}, \quad B = \begin{pmatrix} -t & 0 & 0 & \cdots & 0 & 0 \\ 1 & -t & 0 & \cdots & 0 & 0 \\ 0 & 1 & -t & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & \ddots & -t & 0 \\ 0 & & & & 1 & -t \end{pmatrix}$$

with Q_1, Q_2, Q_3, Q_4 as in §5. We have $\det(f_Z^* - t\text{Id}) = (-1)^{N-1}t^{N-1}\det(A)$, and we evaluate $\det(A)$ to obtain the polynomial given above. \square

§B. Appendix: Invariant Polynomials. We will look for polynomials $P(x) = \sum a_I x^I$ which are invariant in the sense that

$$P \circ f = t \cdot j_f \cdot P \tag{B.1}$$

where $t \neq 0$ is constant, and $j_f = 2x_0(\gamma \cdot x)(\beta \cdot x)^2$ is the Jacobian determinant. If P and Q are solutions to (B.1) with multipliers t_P and t_Q , then $\varphi = P/Q$ is a rational function with the invariance property: $\varphi \circ f = (t_P t_Q^{-1}) \varphi$. If P is a solution to (B.1), then P defines a meromorphic 3-form Ω_P : on the set $x_0 \neq 0$, it is given by $P(1, x_1, x_2, x_3)^{-1} dx_1 \wedge dx_2 \wedge dx_3$. This is invariant in the sense that $f^* \Omega_P = t_P^{-1} \Omega_P$. It follows that $\{P = 0\}$ is an f -invariant surface

which represents the canonical class in \mathbf{P}^3 and its strict transforms are invariant surfaces which represent the canonical classes in Y and Z .

The equation (B.1) can be rewritten as a system of linear equations for the coefficients of the monomials in P . This system can be solved directly for all the maps in §7 and §8. For instance, in §7 ω is a non-real root of unity and $a_0 = a \neq 0$, and we find a solution for $t = \omega^2$:

$$\begin{aligned} P_1 = & (1 - \omega) (a^2 x_0^4 + (1 + a)x_0 x_1 x_2^2 + x_1^2 x_3^2 + a x_1 x_2 x_3^2) \\ & - (2 + \omega) (x_0 x_2^3 + (1 + a)x_0 x_1^2 x_3 + a x_1 x_2^2 x_3 + a x_0^2 x_3^2) \\ & + (1 + 2\omega) (a x_0^2 x_1^2 + a x_0 x_1^2 x_2 + a x_1^2 x_2 x_3 + a x_0 x_2 x_3^2) + a x_0^3 x_1 (1 + a + 2\omega - a\omega) \\ & + (1 - 2a + 2\omega - a\omega) ((1 + a)x_0^2 x_1 x_3 + x_0 x_2^2 x_3) + x_0^2 x_2^2 (1 - a + 2\omega + a\omega) \\ & - (2 - a + \omega + a\omega) ((1 + a)x_0^2 x_1 x_2 + x_0 x_1 x_3^2) + a x_0^3 x_3 (1 - 2a - \omega - a\omega) \\ & + (1 + a)x_0^2 x_2 x_3 (1 + a - \omega + 2a\omega) + a x_0^3 x_2 (2 + a + \omega + 2a\omega) \end{aligned}$$

§C. Appendix: The Rotor Map. Let $g := f_Z^8|_{\Sigma_3}$ denote the rotor map restricted to Σ_3 , which is written in coordinates in (7.3). By factoring the jacobian determinant, we see that there are four exceptional curves.

$$\begin{aligned} \mathcal{C}_1 &= \{ax_0 + \omega x_2 = 0\} \\ \mathcal{C}_2 &= \{ax_0 + ax_1 + \omega x_2 = 0\} \\ \mathcal{C}_3 &= \{a\omega x_0 + x_1 + \omega^2 x_2 = 0\} \\ \mathcal{C}_4 &= \{a\omega x_0^2 + a\omega x_0 x_1 + a\omega x_0 x_2 + \omega^2 x_0 x_2 + x_1 x_2 + \omega^2 x_2^2 = 0\} \end{aligned}$$

Lemma C.1. *If $a \neq \omega^j$ and $a^j \neq \omega^{j \pm 2}$ for all $j \geq 2$ then g is not birationally conjugate to an automorphism.*

Proof. The exceptional curves \mathcal{C}_2 and \mathcal{C}_4 mapped to a three cycle : $g : \mathcal{C}_2 \mapsto [0 : 1 : -a\omega] \mapsto [0 : 1 : -a] \mapsto [0 : 1 : -a\omega^2] \mapsto [0 : 1 : -a\omega]$ and $g : \mathcal{C}_4 \mapsto [1 : 0 - \omega^2] \mapsto [1 : 0 : -\omega] \mapsto [1 : 0 : -1] \mapsto [1 : 0 : -\omega^2]$. For \mathcal{C}_3 we see that $g^j \mathcal{C}_3 = [1 : -\omega^2(\omega/a)^{j-1} : 0]$ for all $j \geq 1$. It follows that these three curves have orbits that do not encounter the indeterminacy locus of g . The remaining exceptional curve \mathcal{C}_1 mapped to $e_1 = [0 : 1 : 0]$, which is indeterminate. We let W be the space obtained by blowing up Σ_3 at e_1 , and we let E_1 be the corresponding exceptional divisor. Under the induced map g_W we have $g_W(E_1) = E_1$ and the orbit of the strict transform of \mathcal{C}_1 remains in E_1 and does not encounter the indeterminacy locus of g_W .

Now if H denote the class of a generic line in W , then $\langle H, E_1 \rangle$ is an ordered basis for $Pic(W)$. The action on Pic is given by the matrix $g_W^* = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$. The largest eigenvalue is $\lambda = (3 + \sqrt{5})/2$ and invariant class is given by $\theta = \lambda H - E_1$. Since $\theta^2 = \lambda^2 - 1 \neq 0$, it follows from [DF, Theorem 5.4] that g is not birationally conjugate to an automorphism. \square

Lemma C.2. *If $a^j = \omega^{j-2}$ for some $j \geq 2$ then g is not birationally conjugate to an automorphism.*

Proof. In case $a^j = \omega^{j-2}$ for some $j \geq 2$, the orbits of three exceptional curves $\mathcal{C}_2, \mathcal{C}_3$, and \mathcal{C}_4 are the same as the previous Lemma. After we blowup e_1 on Σ_3 , the strict transformation of \mathcal{C}_1

mapped to a point of indeterminacy after j -th iteration of g_W . We let W_2 be the space obtained by blowing up W at $g_W^k \mathcal{C}_1$ for $k = 1, \dots, j$ and we let F_k , $1 \leq k \leq j$ be the corresponding exceptional divisors. Under the induced map g_{W_2} , the exceptional line \mathcal{C}_1 is removed and the orbits of remaining three exceptional curves do not encounter the indeterminacy locus of g_{W_2} .

Let $\langle H, F_j, F_{j-1}, \dots, F_1, E_1 \rangle$ be the ordered basis for $Pic(W_2)$. The characteristic polynomial of the action on Pic is given by $t^{j+2} - 4t^{j+1} + 3t^j + t^2 - 2t + 1$. It follows that the dynamical degree is not a Salem number. Thus by [DiF], g is not birationally conjugate to an automorphism. \square

Lemma C.3. *If $a^j = \omega^{j+2}$ for some $j \geq 2$ then g is not birationally conjugate to an automorphism.*

Proof. When $a^j = \omega^{j+2}$, the orbit of \mathcal{C}_3 is different from Lemma C.1, that is $g^{j+1} \mathcal{C}_3 = [1 : -1 : 0]$, which is indeterminate. We let W_3 be the space obtained by blowing up Σ_3 at e_1 and $g^k \mathcal{C}_3$, $1 \leq k \leq j+1$, and we let E_1 and F_k , $1 \leq k \leq j+1$ be the corresponding exceptional divisors. Using the ordered basis $\langle H, F_{j+1}, F_j, \dots, F_1, E_1 \rangle$ for $Pic(W_3)$, we see that the characteristic polynomial of the action on Pic is given by $t^{j+3} - 3t^{j+2} + t^{j+1} + t$. Similarly as in Lemma C.2, the dynamical degree is not a Salem number and therefore g is not birationally conjugate to an automorphism. \square

Lemma C.4. *If $a = \omega$ then g is not birationally conjugate to an automorphism.*

Proof. In this case we see that \mathcal{C}_2 is mapped to a point of indeterminacy under 2 iterations and \mathcal{C}_4 is also mapped to a point of indeterminacy under 3 iterations. After we blowup e_1 , we can check that the orbits of other two remaining exceptional lines does not encounter the indeterminacy locus. After we blow up the orbit of \mathcal{C}_2 and the orbit of \mathcal{C}_4 , we see that the dynamical degree of g is given by the largest root of the polynomial $t^3 - t^2 - 2t - 1$. Again since this number is not a Salem number we have our result. \square

Lemma C.5. *If $a = \omega^2$ then g is not birationally conjugate to an automorphism.*

Proof. If $a = \omega^2$ the each component of g has the same factor $x_0 + x_1 + \omega^2 x_2$. It follows that the restriction of f_Y^8 to Σ_3 is a degree 2 birational map. There are two exceptional lines and both exceptional lines are mapped to points of indeterminacy. After we blowup the points on the orbits of three exceptional lines, we see that the induced map has one exceptional line which is mapped to a point of indeterminacy. Once we blow up this point of indeterminacy, we see that the induced map has no exceptional lines and therefore the induced map is algebraically stable. Furthermore the characteristic polynomial of the action on Pic is $t(1+t)(t-1)^3$ and the action on Pic has 2×2 Jordan block. It follows that the degree of g grows linearly. According to [DiF], we have that g is not birationally conjugate to an automorphism. \square

Lemma C.6. *If $a = 1$ then the degree g grows quadratically.*

Proof. For this case all four exceptional curves are mapped to points of indeterminacy: $g : \mathcal{C}_1 \mapsto e_1$, $\mathcal{C}_2 \mapsto [0 : 1 : -\omega]$, $\mathcal{C}_3 \mapsto [1 : -\omega^2 : 0] \mapsto [1 : -1 : 0]$ and $g : \mathcal{C}_4 \mapsto [1 : 0 : -\omega^2]$. We let Z be the space obtained by blowing up Σ_3 at all five points in the orbit of exceptional curves and we let E_1, Q_2, Q_3, Q_4 , and Q_5 be the corresponding exceptional divisors. Under the induced map g_Z , there is a unique exceptional line which is the strict transformation of \mathcal{C}_1 . We see that $g_Z \mathcal{C}_1$ is a point of indeterminacy of g_Z . By blowing up one more point on E_1 we make the induced map an algebraically stable. Let us denote Q_1 the exceptional divisors corresponding to the point blow ups on E_1 . Let us use $\langle H, F_1, F_2, F_3, F_4, F_5, E_1 \rangle$

as the ordered basis of Pic . The characteristic polynomial of the action on Pic is given by $(t-1)^4(t+1)(t^2+t+1)$ and the matrix representation of the action on Pic has 3×3 Jordan block. It follows that the degree of g grows quadratically. \square

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